Combinatorics of diagrams of permutations

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September 20, 2014
The Grassmannian and the positive Grassmannian

The (real) Grassmannian $\text{Gr}_{n,k}$ of $k$-planes in $\mathbb{R}^n$, represented by $k \times n$ matrices. Decompose in two ways:

- Nicely, as a union of Schubert cells indexed by Grassmannian permutations $w_\lambda$ (those with a unique descent)
- Not nicely, as a union of matroidal cells (where we keep track of precisely which maximal minors are 0)

$$\begin{bmatrix}
1 & 0 & 1 & 1 & 0 & -1 \\
0 & 0 & 0 & -1 & 1 & 1 
\end{bmatrix}$$
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When we restrict to the *nonnegative Grassmannian* $\text{Gr}_{n,k}^\geq$ of $k$-planes represented by matrices with nonnegative maximal minors, the second decomposition becomes nice (“positroids”).
Theorem (Postnikov). The positroidal cells in the Schubert cell indexed by the Grassmannian permutation $w_\lambda$ are in bijection with:

- the regions in the inversion hyperplane arrangement of $w_\lambda$;
- the acyclic orientations of the inversion graph of $w_\lambda$;
- the nonattacking rook placements avoiding a certain diagram of $w_\lambda$;
- the permutations below $w_\lambda$ in strong Bruhat order;
- etc.
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- the regions in the **inversion hyperplane arrangement** of $w_\lambda$;
- the **acyclic orientations** of the **inversion graph** of $w_\lambda$;
- the nonattacking **rook placements** avoiding a certain **diagram** of $w_\lambda$;
- the permutations below $w_\lambda$ in strong Bruhat order;
- etc.

Our Question. What can we say in the case that $w = w_\lambda$ is not necessarily Grassmannian?
Observation: cells of the diagram $O_w$ correspond naturally to co-inversions of $w$
Given $S \subseteq [n] \times [n]$, count placements of nonattacking rooks on $S$:
For a permutation $w \in S_n$, the inversion arrangement $A_w$ is the collection in $\mathbb{R}^n$ of hyperplanes $x_i - x_j = 0$ for each inversion $(i, j)$ in $w$:

- $w = 3142$
- inversions: $\{(1, 2), (1, 4), (3, 4)\}$

$$A_{3142} = 8$$

- $w = 3412$
- inversions: $\{(1, 3), (1, 4), (2, 3), (2, 4)\}$

$$A_{3412} = 14$$
For a permutation $w \in S_n$, the inversion arrangement $A_w$ is the collection in $\mathbb{R}^n$ of hyperplanes $x_i - x_j = 0$ for each inversion $(i, j)$ in $w$.

Classically equivalent: let $G_w$ be the graph on $[n]$ with edge $(i, j)$ for each inversion in $w$. Then $A_w = \#A_w$ is also the number of acyclic orientations of $G_w$: 

\[ G_{3142} \quad \text{and} \quad G_{3412} \]
Our question

Theorem (Postnikov). When $w$ is a Grassmannian permutation, the following objects are in bijection:

- the number of acyclic orientations of the inversion graph of $w$;
- the nonattacking rook placements avoiding a certain diagram of $w$;
- the permutations below $w$ in strong Bruhat order;
- the 0, 1-fillings of $D_w$ avoiding certain patterns;
- etc.

Our Question. What can we say in the case that $w = w_\lambda$ is not necessarily Grassmannian?
Theorem 1 (L–Morales 14). For all permutations \( w \), the number of acyclic orientations of the inversion graph of \( w \) is equal to the number of rook placements avoiding the diagram of \( w \).

Proof idea.
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*Proof idea.* On rook placements, do or do not place a rook (but must stay in permutation-world); on graphs, deletion-contraction (ditto).
Rook placements and acyclic orientations

**Theorem 1** (L–Morales 14). *For all permutations $w$, the number of acyclic orientations of the inversion graph of $w$ is equal to the number of rook placements avoiding the diagram of $w$.*

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Theorem 1 (L–Morales 14). For all permutations $w$, the number of acyclic orientations of the inversion graph of $w$ is equal to the number of rook placements avoiding the diagram of $w$.

Proof idea. On rook placements, do or do not place a rook (but must stay in permutation-world); on graphs, deletion-contraction (ditto).

There is also an alternate proof: show that the chromatic polynomial of the inversion graph is equal to a rook polynomial of the diagram using the ideas of Goldman–Joichi–White RT3, then specialize to $-1$. 

(Strong) Bruhat order is defined by cover relations $w < w \cdot (i, j)$ if $\ell(w) + 1 = \ell(w \cdot (i, j))$. 

\begin{figure}
\centering
\begin{tikzpicture}
  \node at (0,0) {123};
  \node at (1,1) {213};
  \node at (2,2) {231};
  \node at (1,3) {312};
  \node at (0,4) {321};

  \draw (0,0) -- (1,1);
  \draw (1,1) -- (2,2);
  \draw (2,2) -- (1,3);
  \draw (1,3) -- (0,4);
\end{tikzpicture}
\end{figure}
(Strong) Bruhat order is defined by cover relations \( w < w \cdot (i, j) \) if \( \ell(w) + 1 = \ell(w \cdot (i, j)) \).

Lower interval \([e, w]\) is the set of things below \( w \).
**Theorem** (Hultman–Linusson–Shareshian–Sjöstrand 08/09; conjectured by Postnikov). *The number of acyclic orientations of the inversion graph of the permutation $w$ is equal to the size $\# [e, w]$ of the lower order ideal of $w$ in strong Bruhat order if and only if $w$ avoids the permutation patterns $4231, 35142, 42513, 351624$.***
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Corollary 2 (of this and L–M). Ditto if we replace “acyclic orientations of the inversion graph of $w$” with “rook placements avoiding the diagram of $w$.”
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Corollary 2 (of this and L–M). Ditto if we replace “acyclic orientations of the inversion graph of $w$” with “rook placements avoiding the diagram of $w$.”

Remark: there are natural $q$-analogues of rook placements and sizes of Bruhat intervals
The Poincaré polynomial of $w$ is the rank-generating function for $[e, w]: P_{312}(q) = 1 + 2q + q^2$
**Invertible matrices**

Rook placements also have a natural $q$-analogue:

**Theorem** (L–Liu–Morales–Panova–Sam–Zhang). The number of matrices in $GL_n(F_q)$ avoiding $D$ is a $q$-analogue* of the number of rook placements avoiding $D$.

$$4$$ rook placements

$$4$$ invertible matrices

$$(q^2 + 2q + 1)(q - 1)^3 q^3$$ invertible matrices
Theorem (HLSS+LM). The number of rook placements avoiding $O_w$ is equal to $\# [e, w]$ iff $w$ avoids 4231, 35142, 42513, 351624.
The \( q \)-analogue of HLSS

**Theorem (HLSS+LM).** The number of rook placements avoiding \( O_w \) is equal to \( \#[e, w] \) iff \( w \) avoids 4231, 35142, 42513, 351624.

**Theorem 3 (L–Morales).** The number \( M_w(q) \) of matrices in \( \text{GL}_n(F_q) \) avoiding \( O_w \) satisfies

\[
M_w(q) = (q - 1)^n q^{\binom{n}{2}} q^{\ell(w)} P_w(q^{-1})
\]

iff \( w \) avoids 4231, 35142, 42513, 351624.

**Corollary 4 (LM, independently Linusson–Shareshian).** The number \( M_w(q) \) of matrices in \( \text{GL}_n(F_q) \) avoiding \( O_w \) satisfies

\[
M_w(q) = (q - 1)^n q^{\binom{n}{2}} P_w(q)
\]

iff \( w \) is smooth (avoids 4231, 3412).
Theorem (L–Morales). The number $M_w(q)$ of matrices in $GL_n(\mathbb{F}_q)$ avoiding $O_w$ satisfies

$$M_w(q) = (q - 1)^n q^{\binom{n}{2}} q^{\ell(w)} P_w(q^{-1})$$

iff $w$ avoids $4231, 35142, 42513, 351624$.

Proof idea. HLSS show that $\#[e, w]$ obeys certain recurrences when $w$ avoids patterns; these lift to $P_w(q)$. We give related recurrences for matrix counts by considering whether a given entry is $0$ or not, and doing elimination. (But life can be complicated ....)
Final remarks

Corollary: when $w$ avoids patterns, $M_w(q) \in (q - 1)^n \mathbb{N}[q]$. Conjecture: true for all $w$. 
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$M_w(q) \in (q - 1)^n \mathbb{N}[q]$. Conjecture: true for all $w$.

We can also say a little bit about fillings of diagrams.