Matrices with restricted entries and $q$-analogues of permutations

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joint work with
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**Question 1** (Stanley). How many invertible $n \times n$ matrices are there over the field $\mathbb{F}_q$ with all diagonal entries equal to 0?
A pretty enumerative problem

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**Question 1** (Stanley). How many invertible \( n \times n \) matrices are there over the field \( F_q \) with all diagonal entries equal to 0?

For example \( (q = 3) \):

\[
\begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & -1 \\
0 & 1 & 0
\end{bmatrix}
\quad \text{yes}
\quad
\begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & -1 \\
1 & 1 & 0
\end{bmatrix}
\quad \text{no}
\quad
\begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & -1
\end{bmatrix}
\quad \text{no}
\]
Counting invertible matrices with zero diagonal

We can count derangements: build up a permutation entry by entry, get a recursion, guess the answer

\[ \sum_{i=0}^{n} (-1)^i \frac{n!}{i!} \]
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**Theorem 1** (LLMPSZ). The number of invertible \( n \times n \) matrices over \( \mathbb{F}_q \) with all diagonal entries equal to 0 is

\[ q^{(n-1)/2} (q - 1)^n \sum_{i=0}^{n} (-1)^i \binom{n}{i} [n - i]_q! \]

where \([m]_q! = \prod_{j=1}^{m} (1 + q + \ldots + q^{j-1})\)
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**Theorem 1 (LLMPSZ).** *The number of invertible* \( n \times n \) *matrices over* \( \mathbb{F}_q \) *with all diagonal entries equal to 0 is*

\[ q^{(n-1) \cdot \left( \frac{n-1}{2} \right)} (q - 1)^n \sum_{i=0}^{n} (-1)^i \binom{n}{i} [n - i]_q! \]

*where* \([m]_q! = \prod_{j=1}^{m} (1 + q + \ldots + q^{j-1})\)

**Observations:**

- a *q*-analogue of the number of derangements of length *n*
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**Theorem 1 (LLMPSZ).** *The number of invertible* \( n \times n \) *matrices over* \( \mathbb{F}_q \) *with all diagonal entries equal to 0 is*

\[ q^{(n-1)(n-2)}(q - 1)^n \sum_{i=0}^{n} (-1)^i \binom{n}{i} [n - i]_q! \]

*where* \( [m]_q! = \prod_{j=1}^{m} (1 + q + \ldots + q^{j-1}) \)

**Observations:**
- a \( q \)-analogue of the number of derangements of length \( n \)
- a polynomial in \( q \)
Introduction

$q$-analogues

- Enumerative $q$-analogues
- Examples

Polynomiality

Final thoughts
Enumerative $q$-analogues

- Fix $q$ and $n$
- $S$ is a set of minors of the $n \times n$ grid
- $\text{mat}(n, S; q)$ is the number of $n \times n$ invertible matrices over $\mathbb{F}_q$ such that all minors indicated by $S$ are equal to 0
- $P(n, S)$ is the number of $n \times n$ permutation matrices such that all minors indicated by $S$ are equal to 0
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**Theorem 2** (LLMPSZ). For fixed $q, n, S$, we have

$$\text{mat}(n, S; q) \equiv (q - 1)^nP(n, S) \pmod{(q - 1)^{n+1}}.$$ 

(Note that this mod operation is as integers, not as polynomials.)
Examples

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**Example 1.** Given if $S$ is has only $1 \times 1$ minors (a diagram), $\text{mat}(n, S; q)$ is a $q$-analogue of permutations with restricted values. (We’ll use $D$ instead of $S$ for examples of this case.)
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**Example 2.** When $S = \{\{1\} \times \{1\}, \{1, 2\} \times \{1, 2\}, \ldots, \{1, \ldots, n - 1\} \times \{1, \ldots, n - 1\}\}$ then $\text{mat}(n, S; q)$ is a $q$-analogue of indecomposable permutations, i.e., those that don’t fix $\{1, \ldots, k\}$ for any $k$. 

Polynomiality

- Are our $q$-analogues polynomials?
- Young shapes
- Permutation diagrams
- Pattern avoidance

Final thoughts
Are our $q$-analogues polynomials?

Recall Theorem 2:

**Theorem (LLMPSZ).** For fixed $q$, $n$, $S$, we have

$$\text{mat}(n, S; q) \equiv (q - 1)^n P(n, S) \pmod{(q - 1)^{n+1}}.$$ 

**Question 2.** This says nothing about the nature of the function $\text{mat}(n, S; q)$ for fixed $n$, $S$. Is it a polynomial?
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**Answer:** no, even if we restrict to \( 1 \times 1 \) minors (Stembridge).
Are our $q$-analogues polynomials?

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$$\mat(n, S; q) \equiv (q - 1)^n P(n, S) \pmod{(q - 1)^{n+1}}.$$  

**Question 2.** This says nothing about the nature of the function $\mat(n, S; q)$ for fixed $n$, $S$. Is it a polynomial?

Answer: no, even if we restrict to $1 \times 1$ minors (Stembridge). However, in some special cases there may be things to be done!
Young shapes

**Theorem** (Haglund). If $D$ is (the complement of) a Young diagram then $\text{mat}(n, D; q)$ is a polynomial in $q$. In particular, it counts permutation matrices with no 1s in $D$ by number of inversions.
Young shapes

**Theorem** (Haglund). *If* $D$ *is (the complement of) a Young diagram then* $\text{mat}(n, D; q)$ *is a polynomial in* $q$. *In particular, it counts permutation matrices with no 1s in* $D$ *by number of inversions.*

**Theorem 3** (Klein, L., Morales). *If* $D$ *is the complement of a skew Young diagram, ditto.*

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
Permutation diagrams

The *Rothe diagram* of a permutation is what you get by throwing away all the hooks with vertices at the entries of the permutation matrix:
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\[
\begin{bmatrix}
\star \\
\star \\
\star \\
\star \\
\end{bmatrix}
\]

31254 $\rightarrow$
The *Rothe diagram* of a permutation is what you get by throwing away all the hooks with vertices at the entries of the permutation matrix:

\[
31254 \rightarrow \begin{bmatrix}
* \\
* \\
* \\
* \\
*
\end{bmatrix}
\]
Permutation diagrams

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$$31254 \rightarrow \begin{bmatrix}
* & * & * \\
* & * \\
* & * \\
* & *
\end{bmatrix}$$
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\[
31254 \rightarrow \begin{bmatrix}
* & * & * \\
* & * \\
* & * \\
* & * \\
\end{bmatrix}
\]
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\[
\begin{array}{cccc}
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & & \\
\ast & \ast & & & \\
\ast & \ast & & \\
\end{array}
\]

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\[ 31254 \rightarrow \begin{bmatrix}
  * & * & * \\
  * & * & * & * \\
  * & * & * \\
  * & * & * \\
  * & * & * 
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\[
31254 \rightarrow \begin{bmatrix}
\ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast
\end{bmatrix}
\]
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\[
\begin{bmatrix}
\star & \star & \star \\
\star & \star & \star & \star \\
\star & \star & \star \\
\star & \star \\
\star & \star & \star \\
\end{bmatrix}
\]

31254 →
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\[
\begin{pmatrix}
* & * & * \\
* & * & * & * \\
* & * & * & * \\
* & * & * & * & * \\
\end{pmatrix}
\]

31254 →
The *Rothe diagram* of a permutation is what you get by throwing away all the hooks with vertices at the entries of the permutation matrix:

\[
31254 \rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\]
Permutation diagrams

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$$31254 \rightarrow \begin{bmatrix} 0 & 0 \\ 0 & \end{bmatrix}$$

**Question 3** (Postnikov). What can we say about \(\text{mat}(n, D_w; q)\) where \(D_w\) is the diagram of a permutation \(w\)?
Permutation diagrams and pattern avoidance

A permutation $w = w_1 \cdots w_n$ *avoids* a permutation $p = p_1 \cdots p_\ell$ if there is no sequence $1 \leq i_1 < \ldots < i_\ell \leq n$ such that $w_{i_j} < w_{i_k}$ whenever $p_j < p_k$. 
A permutation $w = w_1 \cdots w_n$ avoids a permutation $p = p_1 \cdots p_\ell$ if there is no sequence $1 \leq i_1 < \ldots < i_\ell \leq n$ such that $w_{i_j} < w_{i_k}$ whenever $p_j < p_k$.

- $31524$ does not avoid $2143$ because of $31524$
A permutation $w = w_1 \cdots w_n$ avoids a permutation $p = p_1 \cdots p_\ell$ if there is no sequence $1 \leq i_1 < \ldots < i_\ell \leq n$ such that $w_{i_j} < w_{i_k}$ whenever $p_j < p_k$.

- $31524$ does not avoid $2143$ because of $31524$
- $35124$ avoids $2143$
A permutation \( w = w_1 \cdots w_n \) avoids a permutation \( p = p_1 \cdots p_\ell \) if there is no sequence \( 1 \leq i_1 < \ldots < i_\ell \leq n \) such that \( w_{i_j} < w_{i_k} \) whenever \( p_j < p_k \).

**Theorem** (Lascoux, Schützenberger). The diagram \( D_w \) of \( w \) can be rearranged by row and column swaps to give a Young diagram if and only if \( w \) avoids the permutation pattern 2143.

Consequently (by Haglund’s result), if \( w \in \mathfrak{S}_n \) avoids 2143 then \( \text{mat}(n, D_w; q) \) is a polynomial in \( q \).
A permutation $w = w_1 \cdots w_n$ avoids a permutation $p = p_1 \cdots p_\ell$ if there is no sequence $1 \leq i_1 < \ldots < i_\ell \leq n$ such that $w_{i_j} < w_{i_k}$ whenever $p_j < p_k$.

**Theorem** (Lascoux, Schützenberger). The diagram $D_w$ of $w$ can be rearranged by row and column swaps to give a Young diagram if and only if $w$ avoids the permutation pattern 2143.

**Theorem 4** (Klein, L., Morales). The diagram $D_w$ of $w$ can be rearranged by row and column swaps to give the complement of a skew Young diagram if and only if $w$ avoids the nine patterns 24153, 25143, 31524, 31542, 32514, 32541, 42153, 52143, and 214365.

Consequently (by the previous KLM result), if $w \in \mathfrak{S}_n$ avoids these nine patterns then $\text{mat}(n, D_w; q)$ is a polynomial in $q$. 
A permutation \( w = w_1 \cdots w_n \) avoids a permutation \( p = p_1 \cdots p_\ell \) if there is no sequence \( 1 \leq i_1 < \ldots < i_\ell \leq n \) such that \( w_{i_j} < w_{i_k} \) whenever \( p_j < p_k \).

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**Conjecture 1.** For any fixed \( n, w \in \mathfrak{S}_n \), we have \( \text{mat}(n, D_w; q) \) is a polynomial in \( q \).
Final thoughts
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