The Hurwitz action in real reflection groups

(w/ Vic Reiner)

$G$ a group

$(t_1, \ldots, t_i, t_{i+1}, \ldots, t_m) \in G^m$

$\downarrow \sigma_i$

$(t_1, \ldots, t_{i-1}, t_{i+1}, t_{i+1}, t_{i+1}, \ldots, t_m)$

Obs: preserves product $t_1 \cdots t_m =: c$, so acts on factorizations
Question: connectedness? (Equiv: orbit structure.)

(Background:
- Hurwitz, 1890s, $G = S_n$, 9 transpositions, something to do w/ covering spaces of the Riemann sphere
- Bessis, 2000s, $G = \text{finite real reфнг, ti reфlections}$
  \text{group}
  dual Coxeter systems, eventually something to do w/ $K(\pi, 1)$ Eilenberg-MacLane for complement
  of C hyperplane arrangement of C reфнг gps.)
Thm (Bessis; "dual Matsumoto-Tits lemma"): W a finite real ref
gp of rank \( n \), \( C \in W \) a Coxeter element. The Hurwitz action on
minimal (length \( n \)) factorizations of \( C \)
into reflections is transitive.

Finite real ref
gp: matrix gp

(\( \text{finite, over } \mathbb{R} \)) generated by its subset
of orthogonal reflections thru a hyperplane.

Generators & relations:
\[
\langle s_1, \ldots, s_n | s_i^2 = 1, (s_is_j)^{\circ\circ} = 1 \rangle
\]

"simple refining"

Coxeter elt: \( c = s_1 s_2 \ldots s_n \) (\& conjugates)

E.g.: \( S_n \): perm matrices, \( s_i \): adjacent transpositions,
\( c = (1234 \ldots n) \)

- type B: hyperoctahedral gp / signed perm.

matrices, \( c = \begin{pmatrix} 1 & 2 & 3 & \ldots & n-1 & n \\ 2 & 3 & 4 & \ldots & n & 1 \end{pmatrix} \)

\[
= \begin{pmatrix} 0 & 0 & \ldots & 0 & -1 \\ 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix}
\]

(two-line)

(as a matrix)
$\mathbb{Z}_9 \rightarrow S_3$

$C = (123) = (12)(23)$

$= (23)(13)$

$\sigma_1$
Thm (Bessis; "dual Matsumoto-Tits lemma"): \( W \) a finite real reflection group of rank \( n \), \( c \in W \) a Coxeter element. The Hurwitz action on minimal (length \( n \)) factorizations of \( c \) into reflections is transitive.

Finite real reflection group: matrix group \( \mathfrak{g} \mathfrak{p} \) (finite, over \( \mathbb{R} \)) generated by its subset of orthogonal reflections thru a hyperplane.

Generators & relations:
\[
\langle s_1, \ldots, s_n \mid s_i^2 = 1, (s_i s_j)^3 = 1 \rangle
\]

"Simple"反射s:

\[
\text{Coxeter elt: } c = s_1 s_2 \cdots s_n \text{ (\& conjugates)}
\]

E.g.: \( S_n \) : permutation matrices, \( s_i \) : adjacent transpositions, \( c = (1 \ 2 \ 3 \ 4 \ \cdots \ n) \)

\* Type B: hyperoctahedral group / signed permutation matrices, \( c = (1 \ 2 \ 3 \ 4 \ \cdots \ n-1 \ n) \)

\[
\begin{pmatrix}
1 & 2 & 3 & \cdots & n-2 & n-1 & n \\
0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
\end{pmatrix}
\]

(four-line)

\[
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & 1 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
\end{pmatrix}
\]

(as a matrix)

Thm (L-Reiner) \( W, c, \) ditto. \( w \in W \). The Hurwitz action on factorizations of \( c \) as a product of \( w \) reflections is no transitive and possible.
**Example:** $S_3$

$$C = (123) = (12)(23) \xrightarrow{\sigma_1} (23)(13) \xrightarrow{\sigma_1} (13)(12)$$

**Example:** Type $B_2$

$$W = \left\{ \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}, \begin{bmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{bmatrix} \right\} \text{ (8 elements)}$$

Reflects:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$s_1s_2, \quad s_1 \cdot s_1 \cdot s_1 \cdot s_2$$

$$C = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = s_1 \cdot s_1 \cdot s_1 \cdot s_2$$

$$= s_2 \cdot s_2 \cdot s_1 \cdot s_2$$
An idea from Bessis' proof (and related work):
view \((t_1, \ldots, t_m)\) as a walk
\[ e \rightarrow t_1 \rightarrow t_1 t_2 \rightarrow \ldots \rightarrow t_1 t_2 \ldots t_m \]
on \(W\).

Elements we meet on shortest paths to \(c\) are
"W-non-crossing partitions".

So we cannot use existing tools for \(NC\).

Structure of proof:

Main Thm

\[ \text{Lemma 1: factorizations of arbitrary } w \in W \]

\[ \text{Lemma 2: root circuits are acutely disconnected} \]

But if \(m > n\), we don't need to stay in \(NC(W)\)!
Lemma 1 (L-Reiner) \( W \) a finite real reflection group, 
\( w \) an arbitrary element of \( W \),
\( t = (t_1, \ldots, t_m) \) a factorization of \( w \) into reflections.
The Hurwitz orbit of \( t \) contains a factorization
\[ (t_1', t_2', t_3', t_4', \ldots, t_{2k-1}', t_{2k}', t_{2k+1}', \ldots, t_m') \]
s.t.
\[ t_1' = t_2', t_3' = t_4', \ldots, t_{2k-1}' = t_{2k}' \]
\[ \text{and} \]
\[ (t_{2k+1}', \ldots, t_m') \] is a shortest factorization of \( w \).

Use this to put long factorizations into semi-canonical form;
then use Bessis' results about shortest factorizations of
Cox. en's to finish.
Root systems

W a finite real reftn gp.
Take a pair $\pm \alpha$ of vectors $\perp$ to the reflecting hyperplane of each reflection in $W$.
This is a root system $\Phi = \Phi(W)$.
(Also axioms ...) 

Lemma 2 (L-Reiner)

Suppose $\{x_1, \ldots, x_k\}$ is a circuit (i.e., minimal dependent set) in a root system. Replace $x_i$ by $-x_i$ as needed s.t. dependence

$$c_1x_1 + c_2x_2 + \cdots + c_kx_k = 0$$

has $c_i > 0$. Define acuteness graph $G$ by $V(G) = \{1, \ldots, k\}$,

$$E(G) = \{\{i, j\} : x_i \cdot x_j > 0\}.$$

Then $G$ is disconnected.
Rank (Fiedler): if not restricted to a root system, obtuse-ness graph must be connected, acuteness graph is arbitrary.

How do prove acuteness graphs are disconnected?

\[ R_k^2 = \begin{array}{cc}
& \\
& \\
\end{array} \]

Classical types A, B, C, D, E

Zaslavsky, signed graphs, circuits are understood, very few edges in acuteness graph.

Exceptional types: weeks of computer computations

Lots of nice data produced; interested?