Homework 1: tempered distribution and Fourier transform

Due: Oct 22 in class

Problem 1: Calculate the Fourier transform of the following functions f on $\mathbb{R}$, considered as tempered distributions: denote $\mathbf{1}_{x>0}$ as the function equation to 1 for $x > 0$ and 0 for $x \le 0$, let

(1) $f(x) = \mathbf{1}_{x>0}$;
(2) $f(x) = x\mathbf{1}_{x>0}$;
(3) $f(x) = \sin x$;
(4) $f(x) = x^2 \sin x$.

Problem 2: Define the convolution of a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ with a Schwartz function $\Phi \in \mathcal{S}(\mathbb{R}^n)$ as the function $f * \Phi$ with

$$f * \Phi(x) = f(\Phi(x - \cdot)),$$

where the function $\Phi(x - \cdot)$ is defined as

$$\Phi(x - \cdot)(y) = \Phi(x - y).$$

Show that $f * \Phi \in \mathcal{S}'(\mathbb{R}^n)$ and that

$$\widehat{f * \Phi}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi).$$

Problem 3: In $\mathbb{R}^2$, calculate the Fourier transform of

$$\log |x|.$$

Hint: note the connection with the fundamental solution to Laplace equation in two dimensions.

Problem 4: Solution to the Beltrami equation. In this exercise, you are asked to outline the main steps to solve the Beltrami equation

$$\partial_{\overline{z}} f = \mu(x, y)\partial_z f, \tag{1}$$

where $\mu \in \mathcal{S}(\mathbb{R}^2)$ with $\|\mu\|_{L^\infty} < 1$, and

$$z = x + iy, \qquad \overline{z} = x - iy, \qquad \partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y), \qquad \partial_z = \frac{1}{2}(\partial_x - i\partial_y).$$

Step 1 Find the solution to

$$\partial_{\overline{z}} u = g, \tag{2}$$

for any $g \in \mathcal{S}(\mathbb{R}^2)$. Denote the solution u as $u = C(g)$.

Step 2 Find the expression of the operator T

$$T : \partial_{\overline{z}} g \to \partial_z g \tag{3}$$

for any $g \in \mathcal{S}(\mathbb{R}^2)$, in terms of Fourier transforms. In other words, find the *Fourier multiplier* $m(\xi)$ so that

$$\widehat{Th}(\xi) = m(\xi)\widehat{h}(\xi)$$

and that (3) holds. Show also that $T : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is bounded with

$$\|T\| = 1.$$

Step 3 Now solve solution f to (1) in the form of

$$f = z + h$$

where h satisfies

$$\partial_{\bar{z}} h - \mu T \partial_{\bar{z}} h = \mu. \tag{4}$$

Show by contraction mapping theorem in L^2 that equation (4) has a unique solution $\partial_{\bar{z}} h \in L^2$.

Step 4 h can then be recovered from $\partial_{\bar{z}} h$ using the operator C.

Cross-reference: read the wiki page on Beltrami equation to see the connection to metric on the plane.