# ASYMPTOTICS OF STATIONARY NAVIER STOKES EQUATIONS IN HIGHER DIMENSIONS

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ABSTRACT. We show that the asymptotics of solutions to stationary Navier Stokes equations in 4, 5 or 6 dimensions in the whole space with a smooth compactly supported forcing are given by the linear Stokes equation. We do not need to assume any smallness condition. The result is in contrast to three dimensions, where the asymptotics for steady states are different from the linear Stokes equation, even for small data, while the large data case presents an open problem. The case of dimension $n = 2$ is still harder.

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## 1. INTRODUTION

An important problem in the theory of steady Navier Stokes equations is to understand the asymptotic behavior of solutions at large distances. Consider

$$\left.\begin{array}{rcl} -\Delta u + u\cdot\nabla u + \nabla p & = & f \\ \operatorname{div} u & = & 0 \end{array}\right\} \quad \text{in } R^n \tag{1.1}$$

with an external force $f$. To simplify the technical details, we shall restrict ourselves to a local, regular force in our presentation, i.e., we assume that $f$ is smooth and compactly supported. We consider the case $u(x) \to 0$ as $x \to \infty$. The case when one assumes $u(x) \to u_\infty \neq 0$ as $x \to \infty$ turns out to be easier, see [1] and Section X.8 in [5]. While from the physical point of view the most interesting dimensions are, of course, $n = 3$ and $n = 2$, the higher-dimensional stationary Navier Stokes equation is interesting mathematically. The regularity of steady solutions in higher dimensions have received a lot of attention, as a simpler model for the regularity problem for the time dependent problem. See e.g. [12–14, 16–19, 21]. Here we focus on a different aspect of the high dimensional steady Navier Stokes equations, and study the large distance asymptotics. These problems can be viewed as an analogue of the scattering theory questions in the elliptic setting, and – as we shall see below – have connections with regularity theory. In scattering theory for dispersive equations it can also be the case that some statements are harder to prove in low dimensions, at least if local regularity issues are settled. The reason is similar as in our case: slower decay of the solutions of the linear part of the equations.

The dimension $n$ plays a crucial role in determining the asymptotics, which can already be seen at the linearized level. In the dimension $n = 3$, the fundamental

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solution to the Stokes system decays with a rate $O\left(\frac{1}{|x|}\right)$. Consequently, we can expect that the nonlinearity $u \cdot \nabla u$ to decay with a rate $O\left(\frac{1}{|x|^3}\right)$. Treating $u \cdot \nabla u$ as perturbation and inverting the linear Stokes operator, we can then expect that the contribution of $u \cdot \nabla u$ to the solution $u$ to be of the order $O\left(\frac{1}{|x|}\right)$, which is consistent with the decay of the linear solution. Hence, dimension $n = 3$ appears to be a critical dimension from this point of view. The result in Korolev and Sverak [7] shows that for small $f$, the solution to (1.1) indeed decays with the rate $O\left(\frac{1}{|x|}\right)$. However, the precise asymptotics is different from the one given by the linear Stokes system and is given by an explicit solution found by Landau. More precisely, there exists $\epsilon_* > 0$ such that for a steady state $u$ in $R^3$ satisfying

$$\big|u(x)\big| \leq \frac{\epsilon_*}{1+|x|},$$

one has the following. Denote

$$T_{ij} = p\delta_{ij} + u_i u_j + \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]$$

(the energy momentum tensor), and let

$$b = \int_{|x|=R} T_{ij}\, n_j \, d\sigma,$$

which is independent of $R$ for sufficiently large $R$ as a consequence of

$$\operatorname{div} T = 0$$

outside the support of $f$. The steady state has the asymptotic

$$u = U^b + O\left(\frac{1}{|x|^{1+\beta}}\right) \tag{1.2}$$

for $\beta \in (0,1)$. In the above formula, $U^b$ is the Landau solution, which can be thought of an axi-symmetric solutions of

$$-\Delta U^b + U^b \cdot \nabla U^b + \nabla P^b = b\,\delta(x),$$

with the symmetry axis given by $b$. We refer to [7] for the explicit expression for $U^b$, and more discussion. It appears to be an interesting open problem to determine the precise decay rate of the difference $u - U^b$, and, in particular, whether $\beta$ can be taken as $\beta = 1$ in (1.2).

The result in [7] can be explained at a heuristic level by the fact that the contribution from the nonlinear term $u \cdot \nabla u$ has the same order of magnitude as the typical solution of the linear Stokes system and thus affects the leading term asymptotics. The two dimensional situation is much more complicated still, as the fundamental solution to the linear Stokes system does not even decay, and the above heuristics no longer works. Recently, Guillod [6] put forward some interesting conjectures (in dimension $n = 2$) on the possible asymptotics. We refer the reader to [6] for details.

Although the above analysis, based on linearization, suggests that the steady states in dimension three decays with a rate $O\left(\frac{1}{|x|}\right)$, it remains a major problem to prove this claim once no smallness condition is assumed on $f$. In large data case, the main a priori estimate for a steady solution is the finite Dirichlet energy

$$\int_{R^3} |\nabla u|^2\,dx \le C(f).$$

The finite Dirichlet energy only gives a decay of the order $o\left(\frac{1}{\sqrt{|x|}}\right)$ on average, which is much slower than the expected rate of decay $O\left(\frac{1}{|x|}\right)$. It appears to be an outstanding open problem to prove $O\left(\frac{1}{|x|}\right)$ decay for steady states in dimension 3.

In higher dimensions $n \ge 4$, the problem on the decay of steady states becomes more tractable on a heuristic level, at least. For instance in dimension 4, the a priori estimate

$$\int_{R^4} |\nabla u|^2(x)\,dx \le C(f)$$

already suggests decay of $u$ with the rate $o\left(\frac{1}{|x|}\right)$. This decay rate is already consistent with scale invariance

$$u(x) \to u_\lambda(x) = \lambda u(\lambda x),\ \ p(x) \to p_\lambda(x) = \lambda^2 p(\lambda x),$$

for $\lambda > 0$, in the sense that $u_\lambda$ enjoys uniform bound exterior to $B_1(0)$ for all $\lambda > 1$ assuming that $u$ has the decay $O\left(\frac{1}{|x|}\right)$.

In this short note, we show that the problem is indeed easier in higher dimensions. More precisely, we show that the leading term of the steady state is given by the linear Stokes system:

$$u(x) = G * f(x) + \begin{cases} O(\frac{\log|x|}{|x|^3}) & n = 4 \\ O(\frac{1}{|x|^4}) & n = 5 \\ O(\frac{1}{|x|^5}) & n = 6 \end{cases} \quad \text{as } |x| \to \infty, \tag{1.3}$$

where $G$ is the fundamental solution to the steady Stokes system. Unlike for results in dimension 3, we do not have to assume any smallness condition on $f$.

In principle, the decay problem in higher dimensions $n \ge 7$ is easier (as we shall see below in the proof). However, local regularity could become a problem for large dimensions. Since our method uses a version of $\epsilon$-regularity criteria in the spirit of Scheffer [20] and Caffarelli-Kohn-Nirenberg [3], which is not known for $n \ge 7$, we will only work with dimensions up to 6.

In this paper, we consider general suitable weak solutions to (1.1). There is an important scalar quantity

$$H = |u|^2 + \frac{p}{2},$$

called the "head pressure", which satisfies the scalar equation

$$-\Delta H + u \cdot \nabla H = -|\nabla \times u|^2 - \operatorname{div} f + f \cdot u.$$

Frehse and Růžička proved regularity for weak solutions to (1.1) for which the head pressure $H$ satisfies a suitable maximal principle and established the existence of such solutions for higher dimensions (up to dimension 15 in periodic domains). We refer the reader to the series of works [12–14, 16–19] for details. It is an interesting question if similar asymptotic expansion as in (1.3) for dimensions $n > 6$ can be obtained if we also exploit the special scalar quantity $H$.

## 2. $\epsilon-$REGULARITY CRITERIA

We always assume $n = 4$, 5 or 6. We consider $u \in \dot{H}^1(R^n) \subseteq L^{\frac{2n}{n-2}}(R^n)$, which solves

$$\left.\begin{array}{rcl} -\Delta u + u \cdot \nabla u + \nabla p & = & f \\ \text{div } u & = & 0 \end{array}\right\} \quad \text{in } R^n \tag{2.1}$$

with $f \in C_c^\infty(R^n)$.

We say $u$ is *suitable* if $u$ satisfies the following version of energy inequality:

$$\int_{R^n} |\nabla u|^2(x)\phi(x)\,dx \leq \int_{R^n} \frac{|u|^2}{2}\Delta\phi + \frac{|u|^2}{2} u \cdot \nabla\phi + pu \cdot \nabla\phi + fu\phi\,dx \tag{2.2}$$

for all smooth compactly supported $\phi \geq 0$. Note that this notion is a local one as long as one requires $\phi$ to be supported in a local set. The existence of such solutions are well known, see [9].

The main result we have to use about suitable weak solutions is the following $\epsilon$-regularity theorem.

**Theorem 2.1.** *There exist a sufficiently small $\epsilon_0 > 0$ and positive numbers $c_k$, $k \geq 0$, such that the following statement holds. For any suitable weak solution $u \in H^1(B_1)$ to*

$$\left\{\begin{array}{rl} -\Delta u + u \cdot \nabla u + \nabla p & = 0 \\ \text{div } u & = 0 \end{array}\right. \tag{2.3}$$

*with $\|u\|_{L^3(B_1)} \leq \epsilon_0$ satisfies $u \in C^\infty(B_{1/2})$, and $\|u\|_{C^k(B_{1/2})} \leq c_k$ for all $k \geq 0$.*

The theorem can be proved by classical methods of [10] and [3, 20]. One can for instance follow the argument in [10] for $n = 4, 5$. It is not immediately clear if the same proof works in dimension $n = 6$, due to the lack of compactness in the embedding $H^1(R^6) \to L^3(R^6)$. In this case, one can find a proof in [11](see theorem 2.2 and its proof), which is in the spirit of [3]. In [11], the bounds are not explicitly stated but they are implied in the proof. As remarked in [11], the methods used in proving $\epsilon$-regularity are not likely to work for higher dimensions than 6, due to the fact that the Dirichlet energy can only control $L^3_{\text{loc}}$ norm of $u$ in dimensions up to 6, and in higher dimensions (2.2) no longer makes sense.

We will present a unified proof of this theorem following the approach of [10]. Let us introduce some notations. Denote $(g)_{r_0,x_0}$ as the average of $g$ over the ball $B_{r_0}(x_0)$ and

$$\fint_{B_{r_0}(x_0)} g\,dx = \frac{1}{|B_{r_0}(x_0)|}\int_{B_{r_0}(x_0)} g\,dx.$$

Let

$$Y(g, r_0, x_0) := \left(\fint_{B_{r_0}(x_0)} \left|u - (u)_{r_0,x_0}\right|^3\right)^{\frac{1}{3}}.$$

In the case $x_0 = 0$, we omit the $x_0$ in the above notations. Thus we have, e.g.,

$$(g)_{r_0} = (g)_{r_0,0},\ Y(g, r_0) = Y(g, r_0, 0).$$

We firstly prove

**Lemma 2.1.** *For all $\theta \in (0, 1)$, there exists a sufficiently small $\epsilon_0 = \epsilon_0(\theta) > 0$, and constant $C > 0$ which is independent of $\theta$, such that if $u$ is a suitable weak solution to (2.3) in $B_2$ satisfying*

$$Y(u, 2) \leq \epsilon_0,$$

*and*

$$\left|(u)_2\right| \le 1,$$

*then*

$$Y(u,\theta) \le C\theta\, Y(u,2). \tag{2.4}$$

*Proof:* Suppose the lemma is false, then we can find $\epsilon_i \to 0+$ and suitable weak solutions $u^i$, $p^i$ to (2.3), such that

$$Y(u^i,2) = \epsilon_i,\ \ \left|(u^i)_2\right| \le 1,$$

and that (2.4) does not hold. $p^i$ satisfies in $B_2$

$$-\Delta p^i = \operatorname{div}\operatorname{div}\ \left[(u^i - (u^i)_2) \otimes (u^i - (u^i)_2)\right],$$

and

$$-\nabla p^i = -\Delta u^i + u^i \cdot \nabla\big(u^i - (u^i)_2\big).$$

From the assumption

$$Y(u,2) \le \epsilon,$$

we get that

$$\left\|u^i - (u^i)_2\right\|_{L^3(B_2)} \le \epsilon^i.$$

Standard elliptic estimates imply that modulo constants we have

$$\left\|p^i\right\|_{L^{\frac{3}{2}}(B_{\frac{5}{3}})} \lesssim \epsilon_i.$$

Denote

$$a^i = (u^i)_2$$

and

$$u^i = \epsilon_i\, v^i + a^i,\ \ p^i = \epsilon_i\, q^i.$$

Then $v^i$, $q^i$ verify

$$\left\|v^i\right\|_{L^3(B_{\frac{5}{3}})} \lesssim 1,\ \ \left\|q^i\right\|_{L^{\frac{3}{2}}(B_{\frac{5}{3}})} \lesssim 1, \tag{2.5}$$

and the equation

$$-\Delta v^i + a^i \cdot \nabla v^i + \epsilon_i\, v^i \cdot \nabla v^i + \nabla q^i = 0, \tag{2.6}$$

with

$$\operatorname{div} v^i = 0.$$

We also note that $\left|a^i\right| \le 1$. With some calculations, (2.2) implies that

$$\int_{B_2} \left|\nabla v^i\right|^2 \phi\, dx \le \int_{B_2} \left(\frac{|v^i|^2}{2} + q^i\right) v^i \cdot \nabla\phi + \frac{|v^i|^2}{2}\Delta\phi + \frac{|v^i|^2}{2} a^i \cdot \nabla\phi\, dx, \tag{2.7}$$

for all $\phi \ge 0$ smooth and compactly supported in $B_2$. Take $\phi \in C_c^\infty(B_{\frac{5}{3}})$ with $\phi|_{B_{\frac{4}{3}}} \equiv 1$. From the bounds (2.5) and (2.7), we get that

$$\int_{B_{\frac{4}{3}}} |\nabla v^i|^2\, dx \lesssim 1.$$

Thus bounds on $v^i$, $q^i$ imply that we can assume, by taking a subsequence, that

$$v^i \to v$$

in $L^m(B_{\frac{4}{3}})$ for $m < 3$ and weakly in $H^1(B_{\frac{4}{3}})$, and

$$q^i \to q$$

weakly in $L^{\frac{3}{2}}(B_{\frac{4}{3}})$. We can also assume that the constants $a^i \to a \in R^3$ with $|a| \leq 1$. It is clear that $v^i$, $q^i$ satisfy the equation

$$-\Delta v + a \cdot \nabla v + \nabla p = 0,$$

with

$$\operatorname{div} v = 0$$

and

$$\|v\|_{L^3(B_{\frac{4}{3}})} + \|q\|_{L^{\frac{3}{2}}(B_{\frac{4}{3}})} + |a| \lesssim 1.$$

Elliptic estimates then imply that $v$, $q$ are smooth in $B_1$. In particular,

$$\|v\|_{C^1(B_{\frac{7}{6}})} \lesssim 1.$$

To obtain information on $v^i$ from $v$, let us write

$$v^i = v + \widetilde{v}^i, \;\; q^i = q + \widetilde{q}^i.$$

Clearly $\widetilde{v}^i$, $\widetilde{q}^i$ are uniformly bounded in $L^3(B_{\frac{4}{3}})$ and $L^{\frac{3}{2}}(B_{\frac{4}{3}})$ respectively. In addition, $\widetilde{v}^i \to 0$ in $L^m(B_{\frac{4}{3}})$ for any $m < 3$. $\widetilde{v}^i$, $\widetilde{q}^i$ verify the equation

$$-\Delta\widetilde{v}^i + \left(a^i - a\right) \cdot \nabla v + a^i \cdot \nabla\widetilde{v}^i + \epsilon_i \, v^i \cdot \nabla v^i + \nabla\widetilde{q}^i = 0.$$

Hence

$$-\Delta\widetilde{q}^i = \epsilon_i \operatorname{div}\operatorname{div}\left(v^i \otimes v^i\right).$$

We can decompose $\widetilde{q}^i$ as

$$\widetilde{q}^i = \widetilde{q}^i_1 + \widetilde{q}^i_2,$$

where

$$\widetilde{q}^i_1 = \epsilon_i(-\Delta)^{-1}\operatorname{div}\operatorname{div}\left(v^i \otimes v^i \, \chi_{B_{\frac{4}{3}}}\right),$$

and $\widetilde{q}^i_2$ solves

$$-\Delta\widetilde{q}^i_2 = 0.$$

Then elliptic estimates imply

$$\|\widetilde{q}^i_1\|_{L^{\frac{3}{2}}(R^6)} \lesssim \epsilon_i,$$

and

$$\|\widetilde{q}^i_2\|_{C^1(B_{\frac{7}{6}})} \lesssim 1.$$

Using the smoothness property of $v$, (2.2) implies that

$$\int_{B_{\frac{7}{6}}} \left|\nabla\widetilde{v}^i\right|^2 \phi \, dx \leq \int_{B_{\frac{7}{6}}} \left(\epsilon_i \frac{|v^i|^2}{2} + \widetilde{q}^i\right) \widetilde{v}^i \cdot \nabla\phi - \epsilon_i \, v \cdot \nabla v \, \widetilde{v}^i \, \phi + \\ + \epsilon_i \frac{|v^i|^2}{2} \, v \cdot \nabla\phi - \epsilon_i \, \widetilde{v}^i \cdot \nabla v \, \widetilde{v}^i \, \phi - \left(a^i - a\right) \cdot \nabla v \, \widetilde{v}^i \, \phi + \frac{|\widetilde{v}^i|^2}{2} a^i \cdot \nabla\phi \, dx, \quad (2.8)$$

for nonnegative $\phi \in C^\infty_c(B_{\frac{7}{6}})$ with $\phi|_{B_1} \equiv 1$. We need to show that the right hand side of (2.8) goes to zero as $i \to \infty$. It suffices to consider the term

$$\int_{B_{\frac{7}{6}}} \widetilde{q}^i \, \widetilde{v}^i \cdot \nabla\phi \, dx = \int_{B_{\frac{7}{6}}} \widetilde{q}^i_1 \, \widetilde{v}^i \cdot \nabla\phi \, dx + \int_{B_{\frac{7}{6}}} \widetilde{q}^i_2 \, \widetilde{v}^i \cdot \nabla\phi \, dx.$$

The vanishing of $\widetilde{q}_1^i$ and the smoothness of $\widetilde{q}_2^i$ respectively, together with the fact that $\widetilde{v}^i \to 0$ in $L^m(B_{\frac{4}{3}})$ for any $m < 3$, show that this term vanishes as $i \to \infty$. Therefore, (2.8) implies that

$$\int_{B_1} \left|\nabla \widetilde{v}^i\right|^2 dx \to 0,$$

and consequently

$$v^i \to v, \ in \ L^3(B_1).$$

By the smoothness of $v$, we have

$$\left(\fint_{B_\theta} \left|v - (v)_\theta\right|^3 dx\right)^{\frac{1}{3}} \leq C\theta.$$

Hence, for large $i$,

$$\left(\fint_{B_\theta} \left|v^i - (v^i)_\theta\right|^3 dx\right)^{\frac{1}{3}} \leq C\theta.$$

A contradiction. The lemma is proved.

By scaling and translation invariance, Lemma 2.1 has the following consequence.

**Lemma 2.2.** *Let $\epsilon_0$, $C$ be from Lemma 2.1. Fix $\theta > 0$ sufficiently small so that $C\theta < \frac{1}{2}$. Let $u$, $p$ be a suitable weak solution to (2.3) in $B_{r_0}(x_0)$ with*

$$Y(u, r_0, x_0) \leq \epsilon_0 \, r_0^{-1}, \ \ \left|(u)_{r_0, x_0}\right| \leq r_0^{-1},$$

*then*

$$Y(u, \theta r_0, x_0) \leq \frac{1}{2} Y(u, r_0, x_0). \tag{2.9}$$

Now we can prove Theorem 2.1.

*Proof of Theorem 2.1:* For any $x_0 \in B_{\frac{1}{2}}$. Clearly

$$Y\big(u, \frac{1}{2}, x_0\big) \lesssim \epsilon_0,$$

and

$$\left|(u)_{\frac{1}{2}, x_0}\right| \lesssim \epsilon_0.$$

As long as we choose $\epsilon_0$ sufficiently small, we can apply Lemma 2.2 and obtain that

$$Y\big(u, \frac{\theta}{2}, x_0\big) \leq \frac{1}{2} Y\big(u, \frac{1}{2}, x_0\big).$$

If we can iteratively apply Lemma 2.2 on balls $B_{\frac{\theta^k}{2}}(x_0)$ for $k = 0, 1, \ldots$, we would obtain that

$$Y\big(u, \frac{\theta^k}{2}, x_0\big) \leq \frac{1}{2^k} Y\big(u, \frac{1}{2}, x_0\big).$$

We only need to verify that

$$\left|(u)_{\frac{\theta^k}{2}, x_0}\right| \leq 1,$$

assuming that

$$Y\big(u, \frac{\theta^j}{2}, x_0\big) \leq \frac{1}{2^j} Y\big(u, \frac{1}{2}, x_0\big) \lesssim 2^{-j} \epsilon_0,$$

for $j = 0, 1, \ldots, k-1$. Using the inequality

$$\left|(u)_{\frac{\theta^j}{2}, x_0} - (u)_{\frac{\theta^{j+1}}{2}, x_0}\right| \lesssim Y\big(u, \frac{\theta^j}{2}, x_0\big),$$

for $j = 0, 1, \dots, k-1$, we get that

$$\left|(u)_{\frac{\theta^k}{2},x_0} - (u)_{\frac{1}{2},x_0}\right| \lesssim \epsilon_0.$$

Consequently

$$\left|(u)_{\frac{\theta^k}{2},x_0}\right| \lesssim \epsilon_0.$$

Hence we can iteratively apply Lemma 2.2 and obtain that

$$Y\big(u, \frac{\theta^k}{2}, x_0\big) \le \frac{1}{2^k} Y\big(u, \frac{1}{2}, x_0\big) \lesssim 2^{-k}\epsilon_0, \tag{2.10}$$

for all $k$. As $x_0 \in B_{\frac{1}{2}}$ is arbitrary, (2.10) implies that $u$ is Hölder continuous in $B_{\frac{1}{2}}$. The claimed higher regularity in Theorem 2.1 then follows from standard elliptic estimates.

We shall also need the following variant of Theorem 2.1

**Theorem 2.2.** *Let $\epsilon_0$ be from Theorem 2.1. Let $u$ be a suitable weak solution to (2.3) with $\|u\|_{L^3(B_1)} \le \epsilon_0\,\delta$ for $\delta < 1$. Then there exists constant $C > 1$ which is independent of $u$ and $\delta$, such that $\|u\|_{C(B_{1/4})} \le C\delta$.*

*Proof*: By Theorem 2.1, we know that

$$\|u\|_{C^k(B_{1/2})} \le c_k.$$

Thus $\|u \cdot \nabla u\|_{L^3(B_{1/2})} \le c_1\epsilon_0\delta$. Denote

$$g = u \cdot \nabla u.$$

Then

$$-\Delta u + \nabla p = g,$$

with

$$\|u\|_{L^3(B_{\frac{3}{4}})} \le \epsilon_0\,\delta$$

and

$$\|g\|_{L^3(B_{\frac{3}{4}})} \lesssim \epsilon_0\,\delta.$$

By elliptic estimates, one immediately gets that

$$\|u\|_{W^{2,3}(B_{\frac{1}{2}})} \lesssim \epsilon_0\,\delta.$$

Sobolev embedding then gives

$$\|u\|_{L^m(B_{\frac{1}{2}})} \lesssim \epsilon_0\,\delta$$

for all $m < \infty$. Hence, we have in fact that

$$\|g\|_{L^m(B_{\frac{1}{2}})} \lesssim \epsilon_0\,\delta,$$

for all $m < \infty$. Elliptic estimates immediately imply that

$$\|u\|_{W^{2,m}(B_{\frac{1}{4}})} \lesssim \epsilon_0\,\delta,$$

for all $m < \infty$. By Sobolev embedding, Theorem 2.2 follows.

## 3. Main Theorem and proof

We can now state our main results formally.

**Theorem 3.1.** *Let $u \in \dot{H}^1(R^n) \hookrightarrow L^{\frac{2n}{n-2}}(R^n)$ be a suitable weak solution to equations (2.1). Then*

$$u(x) = G * f(x) + \begin{cases} O(\frac{\log|x|}{|x|^3}) & n = 4 \\ O(\frac{1}{|x|^4}) & n = 5 \\ O(\frac{1}{|x|^5}) & n = 6 \end{cases} \quad \text{as } |x| \to \infty \tag{3.1}$$

*where $G$ is the fundamental solution to linear Stokes equation. That is,*

$$G(x) = -\frac{1}{2n\,\omega_n}\left[\frac{1}{n-2}\frac{I}{|x|^{n-2}} + \frac{x \otimes x}{|x|^n}\right],$$

*where $\omega_n$ is the volume of the unit ball in $R^n$, $I$ is the identity $n \times n$ matix and $x \otimes x$ is the matrix which has $(i,j)$ item $x_i\,x_j$.*

*Proof*: We consider the cases $n = 4$, $n = 5$ and $n = 6$ separately. For $n = 4$, by Sobolev embedding, $u \in \dot{H}^1(R^4) \subset L^4(R^4)$. Take a large $R > 1$ and consider $|x| = 2R$, then direct calculation shows that

$$R^{-1}\int_{B_R(x)} |u|^3\,dy \le \left(\int_{B_R(x)} |u|^4\,dy\right)^{3/4} \to 0,$$

as $R \to +\infty$. Thus by a rescalled version of Theorem 2.1 and 2.2, we get that

$$\big|u(x)\big| = o\left(\frac{1}{|x|}\right), \quad \big|\nabla u(x)\big| = o\left(\frac{1}{|x|^2}\right), \tag{3.2}$$

for $|x| \to +\infty$. Fix small number $\epsilon > 0$ and sufficiently large $R > 1$. Set

$$v(x) = R\,u(Rx). \tag{3.3}$$

(3.2) implies that

$$\big|v(x)\big| + \big|\nabla v(x)\big| \le \epsilon, \quad \text{on } \partial B_1,$$

if $R$ is taken large enough depending on $\epsilon$. In addition,

$$\big|v(x)\big| = o\left(\frac{1}{|x|}\right), \quad \big|\nabla v(x)\big| = o\left(\frac{1}{|x|^2}\right),$$

as $x \to \infty$. By Lemma 3.1 below, we get the desired decay estimate

$$\big|v(x)\big| = O\left(\frac{1}{|x|^2}\right).$$

Using the relation (3.3), we get that

$$\big|u(x)\big| = O\left(\frac{1}{|x|^2}\right).$$

This decay is already sufficient to obtain the precise asymptotics for $u$. To see this, we write

$$\begin{cases} -\Delta u + \nabla p &= f - \text{div } u \otimes u \\ \qquad \text{div } u &= 0 \end{cases} \quad \text{in } R^4.$$

Thus we can write

$$u(x) = G * f(x) - G * (\text{div } u \otimes u)(x).$$

It suffices to prove

$$\left|G * (\text{div } u \otimes u)(x)\right| = O\left(\frac{\log |x|}{|x|^3}\right)$$

under the condition that

$$\left|u(x)\right| = O\left(\frac{1}{|x|^2}\right),$$

as $|x| \to \infty$. This can be verified by direct calculations.

In the case that $n = 5$, by Sobolev embedding, $u \in \dot{H}^1(R^5) \subset L^{10/3}(R^5)$. Suppose $|x| = 2R$ is large, then

$$R^{-2} \int_{B_R} |u|^3 dy \leq R^{-3/2} \left( \int_{B_R(x)} |u|^{10/3}\, dy \right)^{9/10}$$

is small. Thus by rescalled versions of Theorem 2.1 and 2.2, we conclude that $u$ is smooth outside a large ball $B_M$ and

$$\left|u(x)\right| = o\left(\frac{1}{|x|^{3/2}}\right).$$

As before, we can write

$$u = G * f - G * \big(\text{div } u \otimes u\big).$$

Using

$$\left|u(x)\right| = o\left(\frac{1}{|x|^{3/2}}\right)$$

and

$$\left|G(x)\right| \leq \frac{C}{|x|^3},$$

we obtain

$$\left|u(x)\right| = o\left(\frac{1}{|x|^2}\right),$$

which is an improvement of the original estimate. By applying this procedure several times, we obtain the desired estimate.

The case $n = 6$ is almost identical to $n = 5$.

It remains to state and prove

**Lemma 3.1.** *There exists a sufficiently small $\delta > 0$ such that the following statement holds. Let $u$ be a smooth solution to (2.3) in $R^4 \backslash B_1$, satisfying*

$$\left|u(x)\right| \leq \frac{\delta}{|x|}, \ \left|\nabla u(x)\right| \leq \frac{\delta}{|x|^2}, \tag{3.4}$$

*then we have the improved decay estimate*

$$\left|u(x)\right| \lesssim \frac{1}{|x|^2}, \ \left|\nabla u(x)\right| \lesssim \frac{1}{|x|^3}, \tag{3.5}$$

*for $x \in R^3 \backslash B_1$.*

*Proof:* By Theorem 2.1 in [8], we can find a solution $\widetilde{u}(x)$ to (1.1) for $|x| \geq 1$, such that $\widetilde{u}$ is smooth and $\widetilde{u} = u$ on $\partial B_1$. Moreover,

$$\widetilde{u}(x) \lesssim \frac{1}{|x|^2}$$

for $|x| > 1$ and some absolute constant $C > 0$. [1] We claim that

$$u(x) = \widetilde{u}(x), \text{ for } |x| \geq 1.$$

This uniqueness is an easy result in our case as we have a "smallness condition". One can proceed for instance as follows. Denote

$$w = u - \widetilde{u}.$$

Then $w$ satisfies

$$\left\{ \begin{array}{rl} -\Delta w + w \cdot \nabla u + u \cdot \nabla w - w \cdot \nabla w + \nabla p & = 0 \\ \text{div } w & = 0 \end{array} \right. \quad \text{in } R^4 \backslash B_1.$$

Since $w|_{\partial B_1} = 0$, $\nabla w \in L^2$, $w \in L^4$, $w = o(\frac{1}{|x|})$ as $|x| \to \infty$, we obtain by integration by parts:

$$\begin{aligned} \int_{R^4 \backslash B_1} |\nabla w|^2 \, dx &\leq -\int_{R^4 \backslash B_1} (w \cdot \nabla u) w \, dx \\ &= \int_{R^4 \backslash B_1} u \, (w \cdot \nabla w) \, dx \\ &\lesssim \int_{R^4 \backslash B_1} \frac{\delta}{|x|} |w| |\nabla w| \, dx \\ &\lesssim \delta \left( \int_{R^4 \backslash B_1} |\nabla w|^2 \, dx \right)^{1/2} \left( \int_{R^4 \backslash B_1} \frac{|w|^2}{|x|^2} \, dx \right)^{1/2} \\ &\lesssim \delta \int_{R^4 \backslash B_1} |\nabla w|^2 \, dx. \end{aligned}$$

If $\delta$ is sufficiently small, then the above inequality will force $w$ to be identically zero. Thus we obtain that

$$u = \widetilde{u} \quad \text{for } |x| > 1.$$

Since

$$|\widetilde{u}(x)| = O\left( \frac{1}{|x|^2} \right),$$

we also have

$$|u(x)| = O\left( \frac{1}{|x|^2} \right), \text{ as } |x| \to \infty.$$

The lemma is proved.

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[1] We will outline an alternative approach which is somewhat more direct in the appendix.

## 4. Appendix: An alternative approach to Lemma 3.1

The goal of this section is to outline a more direct approach, based more on methods introduced in this note, to Lemma 3.1. Let $u$ be from Lemma 3.1. Take $\overline{u}$ as the solution to the boundary value problem for the steady Stokes equation in the unit ball

$$-\Delta\overline{u} + \nabla\overline{p} = 0, \text{ in } B_1.$$

with $\operatorname{div}\overline{u} = 0$ and $\overline{u}|_{\partial B_1} = u|_{\partial B_1}$. Since $u|_{\partial B_1}$ is smooth, classical estimates for the steady Stokes equations give that $\overline{u}$, $\overline{p}$ are smooth in $\overline{B}_1$. Define

$$U(x) = \begin{cases} u(x) & \text{for } |x| \geq 1, \\ \overline{u}(x) & \text{for } |x| < 1; \end{cases}$$

and

$$P(x) = \begin{cases} p(x) & \text{for } |x| \geq 1, \\ \overline{p}(x) & \text{for } |x| < 1. \end{cases}$$

Then $U$, $P$ verify

$$-\Delta U + U\cdot\nabla U + \nabla P = \big(\overline{u}\cdot\nabla\overline{u}\big)\,\chi_{B_1} + \Sigma,$$

in $R^4$, with $\operatorname{div} U = 0$. In the above, $\Sigma$ is a smooth surface measure supported on $\partial B_1$. Since $u$ is small, both $\overline{u}$ and $\Sigma$ have "smallness condition". Denote

$$g := \big(\overline{u}\cdot\nabla\overline{u}\big)\,\chi_{B_1} + \Sigma.$$

We shall use perturbation argument to find another divergence free solution $V$ to

$$-\Delta V + V\cdot\nabla V + \nabla Q = g, \tag{4.1}$$

with better decay properties than those known for $U$. We can choose the norm

$$\big\|h\big\|_X := \big\|h\big\|_{L^{\frac{5}{2}}(R^4)} + \big\|\nabla h\big\|_{L^2(R^4)},$$

and the space

$$X := \Big\{h:\ \big\|h\big\|_X < \infty\Big\}.$$

We can re-write equation (4.1) as

$$V = G * g - G * \big(V\cdot\nabla V\big). \tag{4.2}$$

We only need to verify that the map

$$V \in X \to G * g - G * \big(V\cdot\nabla V\big) \in X$$

is a contraction mapping in $B_{C\delta} \subseteq X$. This property can be verified by direct calculations. We shall only show that

$$\big\|G * \Sigma\big\|_X \lesssim \delta,$$

assuming that $\Sigma$ is a surface measure supported on the unit sphere with smooth densities and that the density function has $C^m$ norm smaller than $C\delta$ with a sufficiently large $m$. By the decay property of $G$, we only need to show that

$$\big\|\nabla G * \Sigma\big\|_{L^2(R^4)} \lesssim C\delta. \tag{4.3}$$

By the property of Fourier transform of smooth measures supported on sphere, we see that

$$\left|\mathcal{F}(\Sigma)(\xi)\right| \lesssim \frac{\delta}{(1+|\xi|)^{\frac{3}{2}}},$$

where we use the notation $\mathcal{F}(h)$ to denote the Fourier transform of $h$. In combination of the fact that

$$\left|\mathcal{F}(G)(\xi)\right| \lesssim \frac{1}{|\xi|^2},$$

we get that

$$\left|\mathcal{F}(G*\Sigma)(\xi)\right| \lesssim \frac{\delta}{|\xi|^2\,(1+|\xi|)^{\frac{3}{2}}},$$

Hence,

$$\left\| \, |\nabla|^s G*\Sigma \right\|_{L^2(R^4)} \lesssim \delta,$$

for $0<s<\frac{3}{2}$. (4.3) thus follows. $V \in X$ implies that $V \in L^3(R^4)$. The improved decay breaks the scaling and now we can treat the nonlinearity $V\cdot\nabla V$ as perturbations when we consider decay. Indeed, from $V \in L^3$, rescaled version of Theorem 2.1 and 2.2 already implies that

$$\left|V(x)\right| \lesssim \frac{1}{|x|^{\frac{4}{3}}},$$

for large $x$. Treating $V\cdot\nabla V$ as perturbation and following the arguments as in the proof of Theorem 3.1, we can conclude that

$$\left|V(x)\right| \lesssim \frac{\delta}{1+|x|^2}.$$

The same argument as in the proof of Lemma 3.1 shows that

$$U = V.$$

Hence $U$ has the same decay.

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