Syzygies of Random Monomial Ideals

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December 10, 2017

arXiv:1706.01488
Definition
Define a *Random Stanley-Reisner Ideal* by taking the Stanley-Reisner ideal associated with a random flag complex, denoted by $\Delta(n, p)$. 

Example

\begin{align*}
3 & \quad 5 \\
5 & \quad 6
\end{align*}
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Example

\[
\begin{array}{ccc}
6 \\
4 & 3 & 5 \\
1 & 2
\end{array}
\]
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Example

```
    6
   / \
  4   3   5
 /   /   /
1   2   5
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Random Monomial Ideals

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Example

$$I_\Delta = (x_1x_3, x_1x_5, x_1x_6, x_2x_4, x_2x_5, x_2x_6, x_4x_5, x_4x_6)$$
Definition
Let $k$ be a field and $M$ a module, then the betti numbers are

$$\beta_{i,j}(M, k) = \dim Tor_i(M, k)_j$$

These are then usually displayed as a table where the $p$-th column and $q$-th row is the $\beta_{p,p+q}$ entry

$$
\begin{array}{cccc}
\beta_{0,0} & \beta_{1,1} & \beta_{2,2} & \cdots \\
\beta_{0,1} & \beta_{1,2} & \beta_{2,3} & \cdots \\
\vdots & \vdots & \vdots & \ddots \\
\end{array}
$$
Theorem (Ein-Lazarsfeld, 2012)

Let $X$ be a smooth, $d$-dimensional projective variety and let $A$ be a very ample divisor on $X$. For any $n \geq 1$, let $S_n$ be the homogeneous coordinate ring of $X$ embedded by $nA$. For each $1 \leq k \leq d$, $\rho_k(S_n) \to 1$ as $n \to \infty$.

Definition

Let $M$ be a module over a polynomial ring, then we define a “density” for the $k$-th row of the betti table as follows

$$
\rho_k(M) := \frac{\#\{i \in [0, \text{pdim}(M)] \text{ where } \beta_{i,i+k}(M) \neq 0\}}{\text{pdim}(M) + 1}.
$$
Question
What other families have the property that $\rho_k(l_n) \to 1$

- Integral Varieties (Zhou ’14)
- Arithmetically CM (Ein Erman Lazarsfeld ’16)
- Iterated subdivision of Stanley Resiner rings (Conca Juhnke-Kubitzke Welker ’14)
Asymptotic Syzygies

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- Integral Varieties (Zhou '14)
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- Random Stanley Reisner Ideals
Nonvanishing

Theorem (Erman-Y., 2017)

Fix $i, j$ with $1 \leq i$ and $i + 1 \leq j \leq 2i$ and let $r := j - i - 1$. Fix some constant $0 < \epsilon \leq \frac{1}{2}$ and let $\Delta \sim \Delta(n, p)$.

1. If $\frac{1}{n^{1/r}} \ll p \leq \epsilon$ then $\mathbb{P}[\beta_{i,j}(S/I_\Delta) \neq 0] \to 1$.
2. If $p \ll \frac{1}{n^{1/r}}$ then $\mathbb{P}[\beta_{i,j}(S/I_\Delta) = 0] \to 1$.

Corollary (Erman-Y., 2017)

Fix some $r \geq 1$. Let $\Delta \sim \Delta(n, p)$ with $\frac{1}{n^{1/r}} \ll p \ll 1$. For each $1 \leq k \leq r + 1$, we have

$$\rho_k(S/I_\Delta) \to 1$$

in probability.
Theorem (Erman-Y., 2017)

Fix $i, j$ with $1 \leq i$ and $i + 1 \leq j \leq 2i$ and let $r := j - i - 1$. Fix some constant $0 < \epsilon \leq \frac{1}{2}$ and let $\Delta \sim \Delta(n, p)$.

1. If $\frac{1}{n^{1/r}} \ll p \leq \epsilon$ then $\mathbb{P}[\beta_{i,v}(S/I_\Delta) \neq 0] \to 1$.

2. If $p \ll \frac{1}{n^{1/r}}$ then $\mathbb{P}[\beta_{i,v}(S/I_\Delta) = 0] \to 1$. 

Individual entries are 0
Sketch of Proof

For the first statement it suffices to show that $E[\beta_{i,j}]$ goes to infinity. For that we’ll apply Hochster’s formula.

**Theorem (Hochster’s Formula)**

*For $\Delta$ a simplicial complex and $I_\Delta$ the associated Stanley-Reisner Ideal*

$$
\beta_{i,j}(S/I_\Delta) = \sum_{|\alpha| = j} \dim \tilde{H}_{i-j-1}(\Delta|_\alpha)
$$
We’ll restrict specifically to the case where $j = 2i$ and $i = r + 1$. To find an lower bound on $\mathbb{E}[\beta_{i,j}]$, we consider the following particular flag complex

\[ \Diamond_r = r\text{-fold suspension of 2-points} \]

\[ \begin{align*} r = 0 & \quad r = 1 & \quad r = 2 \end{align*} \]
Lemma
\( \tilde{H}_r(\diamond_r) = \mathbb{Z} \) and \( \diamond_r \) has the fewest vertices of any flag complex with \( r \)-th homology.

Then

\[
\mathbb{E}[\beta_{i,j}(S/I_\Delta)] \geq \mathbb{E}[\# \text{ of } \diamond_r \text{ as subcomplexes}]
= Cn^{2r+2} p^{r(2r+2)}(1 - p)^{r+1}
= C(n p^r)^{2r+2} (1 - p)^{r+1}
\]

Then if \( \frac{1}{n^{1/r}} \ll p \ll 1 \), we get that

\[
\mathbb{E}[\beta_{i,j}(S/I_\Delta)] \to \infty
\]
There is another asymptotic behavior that we observed, in this case inspired by a result of Ein, Erman, and Lazarsfeld.

**Theorem (Ein-Erman-Lazarsfeld 2012)**

For $C$ a smooth projective curve, with $C \xrightarrow{|A_n|} \mathbb{P}^N$, where the degrees of the $A_n$ are increasing, then the first row of the betti table of converges to a binomial.
Theorem (Erman-Y., 2017)

Fix a constant $0 < c < 1$ and let $\Delta \sim \Delta(n, \frac{c}{n})$ be a random flag complex. If $\{i_n\}$ is an integer sequence satisfying $i_n = n/2 + o(n)$, and if $C := \frac{1-c}{2}$, then

$$\frac{\beta_{i_n, i_n+1}(S/I_{\Delta})}{Cn\binom{n}{i_n}} \longrightarrow 1 \quad \text{in probability.}$$
Theorem (Erman-Y., 2017)

Fix a constant $0 < c < 1$ and let $\Delta \sim \Delta(n, \frac{c}{n})$ be a random flag complex. If $\{i_n\}$ is an integer sequence satisfying $i_n = n/2 + o(n)$, and if $C := \frac{1-c}{2}$, then

$$\frac{\beta_{i_n, i_n+1} (S/I_{\Delta})}{C^n \binom{n}{i_n}} \xrightarrow{\text{in probability}} 1$$
Again I’ll provide a lower bound, the upper bound proceeds similarly.
Because we took \( p = c/n \) for \( 0 < c < 1 \) we can assume that our graph is always a collection of unicyclic components.

\[
\beta_{i,i+1}(S/I_\Delta) = \sum_{|\alpha|=i} \tilde{H}_0(\Delta|\alpha) \\
= \sum_{|\alpha|=i} i - E(\Delta|\alpha) + C(\Delta|\alpha) \\
\geq \binom{n}{i} \cdot i - \sum_{|\alpha|=i} E(\Delta|\alpha) \\
= \binom{n}{i} \cdot i - \binom{n}{i-2} E(\Delta) \\
\sim \binom{n}{i} (Cn)
\]
Theorem (Erman-Y., 2017)

For any $k \geq 1$, and any $p$ satisfying $\frac{1}{n^{2/3}} \ll p \ll \left( \frac{\log(n)}{n} \right)^{2/(k+3)}$ we have that $\frac{\text{codim}(S/I_\Delta)}{\text{pdim}(S/I_\Delta)} \to 1$ in probability, yet the probability that $S/I_\Delta$ is Cohen-Macaulay goes to 0.
Other Applications of Randomness

- Random Toric Surfaces (Y. 2016)
- Other models of Random Monomial Ideals (De Loera, Petrovíc, Silverstein, Stasi, Wilburne 2017)
Thank You