Math 5385 - Spring 2019
Problem Set 11

Submit solutions to **four** of the following problems.

1. In this problem we will consider the addition and multiplication operations in the quotient ring $\mathbb{R}[x]/\langle x^2+1\rangle$.
   (a) Show that every $f \in \mathbb{R}[x]$ is congruent module $I = \langle x^2+1\rangle$ to some $ax+b$ where $a, b \in \mathbb{R}$.
   (b) Construct formulas for the addition and multiplication rules in $\mathbb{R}[x]/\langle x^2+1\rangle$ using these polynomials as the standard representatives for classes.
   (c) Do we know another way to describe $\mathbb{R}[x]/\langle x^2+1\rangle$?

2. 5.2.10 This problem illustrates one important use of nilpotent elements in rings. Let $R = \mathbf{k}[x]$ and let $I = \langle x^2\rangle$. (**Note.** Nilpotent means that $f^n = 0$ for some $n > 0$)
   (a) Show that $[x]$ is a nilpotent element in R/I and find the smallest power of $[x]$ equal to 0.
   (b) Show that every class in R/I has a unique representative of the form $b + a\epsilon$ where $\epsilon := [x]$.
   (c) Given $b+a\epsilon \in R/I$, we can define the mapping $R \to R/I$ by substituting $x = b+a\epsilon$ into $f(x)$. Show that
   $$f(b+a\epsilon) = f(b) + a \cdot f'(b)\epsilon$$
   where f' is the formal derivative of f
   (d) Suppose $\epsilon = [x] \in R/\langle x^3\rangle$ derive a formula analogous to (3) for $f(b+a\epsilon)$

3. Let $I = \langle y + x^2 - 1, xy - 2y^2 + 2y\rangle \subset \mathbb{R}[x, y]$. Note this is the example after Proposition 5.3.4 and Formula 5.3.2 gives a Grobner basis for it.
   (a) Construct a vector space isomorphism $\mathbb{R}[x, y]/I \cong \mathbb{R}^4$.
   (b) Using the Grobner basis from 5.3.2, compute a multiplication table for the elements $\{[1], [x], [y], [y^2]\}$ writing each product as a linear combination of these four classes.
   (c) is $\mathbb{R}[x, y]/I$ a field, why or why not.

4. Suppose that $I \subset \mathbb{C}[x_1, \ldots, x_n]$ is a radical ideal with Grobner basis $g_1, \ldots, g_n$ with $LM(g_i) = x_i^{m_i}$ for some $m_i > 0$. Prove that $V(I)$ contains exactly $m_1 \cdot m_2 \cdots m_n$ points.

5. This excercise will outline a proof that $V = V(y^5 - x^2) \subset \mathbb{R}^2$ is not isomorphic to $\mathbb{R}$ as a variety (Problem 5.4.14 from [IVA]).
   (a) Show that every polynomial in $\mathbb{R}[V]$ can be uniquely represented by a polynomial of the form $a(y) + b(y)x$ for $a, b \in \mathbb{R}[y]$.
   (b) Express the product $(a + bx)(a' + b'x)$ in $\mathbb{R}[V]$ for $a, b, a', b' \in \mathbb{R}[y]$.
   (c) Show that there is no ring isomorphism $\alpha : \mathbb{R}[t] \to \mathbb{R}[V]$.