Math 5385 - Spring 2019
Problem Set 2

Submit solutions to **all** of the following problems.

1. (a) Show that the variety $V(x^2 - 2)$ is empty over $\mathbb{Q}$ but not over $\mathbb{F}_7$
   (b) Write $V(x^2 - 2)$ as a set over $\mathbb{F}_7$
2. (a) Show that $X = \{(x, x) \mid x \in \mathbb{R}, x \neq 1\} \subset \mathbb{R}^2$ is not an affine variety.
   Hint. If $f \in \mathbb{R}[x, y]$ vanishes on X, then prove that $f(1, 1) = 0$. Consider $g(t) := f(t, t)$.
   (b) Show that $Y = \{(x, y) \in \mathbb{R}^2 \mid y > 0\} \subset \mathbb{R}^2$ is not an affine variety.
3. Consider the set $U(1) := \{z \in \mathbb{C} \mid z\bar{z} = 1\}$.
   (a) If we identify $\mathbb{C}$ with $\mathbb{R}^2$, then show that $U(1)$ is an affine subvariety of $\mathbb{R}^2$.
   (b) Prove that $U(1)$ is not an affine subvariety of $\mathbb{C}^1$.
4. Consider the map $\sigma\colon \mathbb{A}^3 \to \mathbb{A}^6$ defined by $(x, y, z) \mapsto (x^2, xy, xz, y^2, yz, z^2)$. Let a, b, c, d, e, f denote the corresponding coordinates on $\mathbb{A}^6$.
   (a) Show that the image of σ satisfies the equations given by the 2-minors of the symmetric matrix
   $$\Omega = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}.$$
   (b) Compute the dimension of the vector space V in $S = \Bbbk[a, b, c, d, e, f]$ spanned by these 2-minors.
   (c) Show that every homogeneous polynomial of degree 2 in S vanishing on the image of σ is contained in V.