Math 5385 - Spring 2019
Problem Set 3

Submit solutions to **four** of the following problems.

1. Consider the curve, called a *strophoid*, with the trigonometric parametrization given by

$$x = a\sin(t) \qquad y = a\tan(t)\,(1+\sin(t))\,,$$

   where a is a constant.

   (a) Find the implicit equation in x and y that describes the strophoid. **Hint.** Try solving for sin(t) and cos(t), be careful that you get an equation that exactly describes the strophoid.

   (b) Find a rational parametrization of the strophoid.

2. (a) Prove the equality of the ideals $\langle x+xy, y+xy, x^2, y^2\rangle = \langle x, y\rangle$.

   (b) Prove that $V(x+xy, y+xy, x^2, y^2) = V(x, y)$ (Without directly using the equality of ideals).

3. An ideal $I \subseteq k[x_1, \ldots, x_n]$ is said to be *radical* if for any $f \in k[x_1, \ldots, x_n]$, whenever $f^m \in I$, then also $f \in I$.

   (a) Prove that for an affine variety $V \subseteq k^n$, $I(V)$ is always a radical ideal.

   (b) Prove that $\langle x^2, y^2\rangle$ is not a radical ideal. This implies that $\langle x^2, y^2\rangle \neq I(V)$ for any variety $V \subseteq k^2$.

4. Here we study the *consistency problem* from §1.2 in the one-variable case. Given $f_1, \ldots, f_s \in \mathbb{k}[x]$, this asks if there is an algorithm to decide if $V(f_1, \ldots, f_s)$ is nonempty. You will show that the answer is yes when $\mathbb{k} = \mathbb{C}$.

   (a) Let $f \in \mathbb{C}[x]$ be a nonzero polynomial. Use Theorem 1.1.7 to show that $V(f) = \varnothing$ if and only if f is constant.

   (b) If $f_1, \ldots, f_s \in \mathbb{C}[x]$, prove $V(f_1, \ldots, f_s) = \varnothing$ if and only if $\gcd(f_1, \ldots, f_s) = 1$.

   (c) Describe (in words, not pseudocode) an algorithm for determining whether or not $V(f_1, \ldots, f_s) \subseteq \mathbb{A}^n(\mathbb{C})$ is nonempty.

   When $\mathbb{k} = \mathbb{R}$, the consistency problem is much more difficult. It requires giving an algorithm that tells whether a polynomial $f \in \mathbb{R}[x]$ has a real root.

5. Fix a monomial order $>$ on $k[x_1, \ldots, x_n]$

   (a) For $f \in k[x_1, \ldots x_n]$ and m a monomial, show that $\mathrm{LT}(f \cdot m) = \mathrm{LT}(f) \cdot m$

   (b) Is $LT(f \cdot g) = LT(f) \cdot LT(g)$? Prove or provide a counter example.