

**Math 5385 - Spring 2019**  
**Problem Set 4**

Submit solutions to **four** of the following problems.

1. Suppose that  $\mathbb{k}$  is an infinite field. Let  $X \subset \mathbb{A}^3(\mathbb{k})$  be the set  $X = \{(t, t^2, t^3) \mid t \in \mathbb{A}^1(\mathbb{k})\}$ .
  - (a) Use the parametrization of  $X$  to show that  $z^2 - x^4y$  vanishes at every point.
  - (b) Find a representation  $z^2 - x^4y = h_1(y - x^2) + h_2(z - x^3)$ , where  $h_1, h_2 \in \mathbb{k}[x, y, z]$ .
  - (c) Use the division algorithm to show that  $I(X) = \langle y - x^2, z - x^3 \rangle$ .
  
2. Let  $I = \langle x^{u_1}, \dots, x^{u_p} \rangle$  and  $J = \langle x^{v_1}, \dots, x^{v_q} \rangle$  be two monomial ideals in  $S = \mathbb{k}[x_1, \dots, x_n]$ .
  - (a) If  $x^w$  is a monomial in  $S$ , then prove that the ideal  $(I : x^w) := \{f \in S \mid fx^w \in I\}$  is generated by the monomials of  $x^{u_i} / \gcd(x^{u_i}, x^w)$  for  $1 \leq i \leq p$ .
  - (b) Show that  $I \cap J$  is generated by monomials  $\text{lcm}(x^{u_i}, x^{v_j})$  for  $1 \leq i \leq p$  and  $1 \leq j \leq q$ .
  
3. Assume that  $x_1 > x_2 > \dots > x_n$ . Show that the following properties characterize the monomial orders  $>_{\text{lex}}$  and  $>_{\text{grevlex}}$  among all monomial orders on  $S = \mathbb{k}[x_1, \dots, x_n]$ .
  - (a) If  $\text{LT}_{\text{lex}}(f) \in \mathbb{k}[x_i, \dots, x_n]$  for some  $1 \leq i \leq n$ , then  $f \in \mathbb{k}[x_i, \dots, x_n]$ .
  - (b) The monomial order  $>_{\text{grevlex}}$  refines the partial order given by total degree and, for homogeneous  $f$ , the condition  $\text{LT}_{\text{grevlex}}(f) \in \langle x_i, \dots, x_n \rangle$  for some  $1 \leq i \leq n$  implies that  $f \in \langle x_i, \dots, x_n \rangle$ .
  
4. Let  $M$  be an  $(m \times n)$ -matrix with nonnegative real entries and let  $r_1, \dots, r_m$  denote the rows of  $M$ . Assume that  $\ker(M) \cap \mathbb{Z}^n = \{0\}$ . Define a binary relation  $>_M$  on the monomials in  $S = \mathbb{k}[x_1, \dots, x_n]$  as follows:  $x^u >_M x^v$  if there is a  $\ell \leq m$  such that  $u \cdot r_i = v \cdot r_i$  for all  $1 \leq i \leq \ell - 1$  and  $u \cdot r_\ell > v \cdot r_\ell$ .
  - (a) Show that  $>_M$  is a monomial order on  $S = \mathbb{k}[x_1, \dots, x_n]$ .
  - (b) If  $M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ , then  $>_M$  equals  $>_{\text{grevlex}}$  on  $\mathbb{k}[x, y, z]$ .
  - (c) If  $I$  is the  $(n \times n)$ -identity matrix, then show that  $>_{\text{lex}}$  equals  $>_I$ .
  
5. A ring  $R$  satisfies the *descending chain condition* if any descending sequence of ideals in  $R$  stabilizes, i.e.  $I_1 \supset I_2 \supset I_3 \supset \dots$  implies  $I_j = I_{j+1}$  for  $j \gg 0$ . Such a ring is *Artinian*.
  - (a) Show that  $\mathbb{Z}/n\mathbb{Z}$  and  $k[x]/\langle x^n \rangle$  are Artinian.
  - (b) Show that  $\mathbb{Z}$  and  $k[x]$  are not Artinian.
  - (c) Show that every prime ideal in an Artinian ring is maximal.  
**Note.** We'll get to prime ideals properly later in class, but the definition is an ideal is prime if for any  $a \cdot b \in I$ , we have  $a \in I$  or  $b \in I$   
**Hint.** If  $J \supset I$ , then consider  $J \supset J^2 \supset J^3 \supset \dots \supset I$ .