Math 5385 - Spring 2019
Problem Set 5

Submit solutions to **three** of the following problems. If you use M2 to solve 3 or 4, please show the intermediate steps.

1. Fix the lexicographic order order on $S = \mathbb{k}[x_1, \ldots, x_n]$ with $x_1 > \cdots > x_n$. Let $A = [a_{i,j}]$ be an $m \times n$ matrix with entries in $\mathbb{k}$ and let $f_i = a_{i,1}x_1 + \cdots + a_{i,n}x_n$ be the linear polynomials in S determined by the rows of A. Suppose that $B = [b_{i,j}]$ is the row-reduced echelon matrix determined by A and let $g_1, \ldots, g_r$ be the linear polynomials determined by the nonzero rows in B.
   (a) Prove that $\langle f_1, \ldots, f_m \rangle = \langle g_1, \ldots, g_r \rangle$.
   (b) Show that $g_1, \ldots, g_r$ form a Gröbner basis of $\langle f_1, \ldots, f_m \rangle$. (See hint on p.96.)
   (c) Explain why $g_1, \ldots, g_r$ is the reduced Gröbner basis.

2. Consider the ideal $I = \langle x^2 + y^2 + z^2 + 2, 3x^2 + 4y^2 + 4z^2 + 5 \rangle$. Let $X = V(I)$, let $\pi \colon \mathbb{A}^3(\mathbb{k}) \to \mathbb{A}^2(\mathbb{k})$ be the projection given by $(x, y, z) \mapsto (y, z)$, and let $J = I \cap \mathbb{k}[y, z]$.
   (a) If $\mathbb{k} = \mathbb{C}$, then prove that $V(J) = \pi(X)$.
   (b) If $\mathbb{k} = \mathbb{R}$, then prove that $X = \varnothing$ and $V(J)$ is infinite. Hence, $V(J)$ may be much larger than the smallest affine variety containing $\pi(X)$ when the field is not algebraically closed.

3. Use elimination to solve the system:

$$0 = x^2 + 2y^2 - y - 2z, \qquad 0 = x^2 - 8y^2 + 10z - 1, \qquad 0 = x^2 - 7yz.$$

   How many solutions are there in $\mathbb{A}^3(\mathbb{R})$; how many are there in $\mathbb{A}^3(\mathbb{C})$?

4. Let $\mathbb{k}$ be an infinite field.
   (a) Consider

$$A := \begin{bmatrix} 1&1&1&1&0&0&0&0 \\ 0&0&0&0&1&1&1&1 \\ 1&1&0&0&1&1&0&0 \\ 0&0&1&1&0&0&1&1 \\ 1&0&1&0&1&0&1&0 \\ 0&1&0&1&0&1&0&1 \end{bmatrix}.$$

   Find a generating set for the toric ideal I_A, which is the ideal for the smallest variety containing the image of the map $\varphi \colon \mathbb{A}^6(\mathbb{k}) \to \mathbb{A}^8(\mathbb{k})$ given by

$$\varphi(t_1, \ldots, t_6) \mapsto (t_1t_3t_5, t_1t_3t_6, t_1t_4t_5, t_1t_4t_6, t_2t_3t_5, t_2t_3t_6, t_2t_4t_5, t_2t_4t_6).$$

   (b) Find the equations for the image of the rational map $\rho \colon \mathbb{A}^4(\mathbb{k}) \to \mathbb{A}^6(\mathbb{k})$ defined by

$$\rho(x_1, x_2, x_3, x_4) = \left(\frac{1}{x_1x_2}, \frac{1}{x_1x_3}, \frac{1}{x_1x_4}, \frac{1}{x_2x_3}, \frac{1}{x_2x_4}, \frac{1}{x_3x_4} \right).$$