Math 5385 - Spring 2019
Problem Set 6

Submit solutions to **three** of the following problems. M2 is suggested for problem 1, but directly use grobner bases, and don't use the remainder feature in M2.

1. Determine whether $f = xy^3 - z^2 + y^5 - z^3$ is in the ideal $I = \langle -x^3 + y, x^2y - z\rangle$.

2. Given $G = (g_1, \ldots, g_s) \in (\mathbf{k}[x_1, \ldots, x_n])^s$ Define a syzygy on G (not on the leading terms) to be an s-tuple $S = (h_1, \ldots, h_s) \in (\mathbf{k}[x_1, \ldots, x_n])^s$ such that $\sum_{i=1}^{s} h_i g_i = 0$
   (a) Show that if $G = (x^2 - y, xy - z, y^2 - xz)$, then $(z, -y, x)$ defines a syzygy on G
   (b) Find another syzygy on G other than the one from part (a)
   (c) Show that if S,T are syzygies on G, and $g \in \mathbf{k}[x_1, \ldots, x_n]$, then $S + T$ and gS are also syzygies on G

3. (See Exercise 3.4.11 for a picture) Consider the curve given by the equation $(x^2 + y^2)^3 = 4x^2y^2$
   (a) Show that most lines through the origin meet the curve with multiplicity 4 at the origin.
   (b) Find the lines through the origin that meet the curve with multiplicity > 4. Give a geometric explanation for the multiplicities.

4. (a) Show that if $f \in \mathbf{k}[x_1, \ldots, x_n]$ is irreducible and divides $h_1 \cdots h_s$ then $f | h_i$ for some i
   (b) Prove that for $f \in \mathbf{k}[x_1, \ldots, x_n]$ with $f = g_1 \cdots g_r = h_1 \cdots h_s$ with g_i and h_i irreducible, then up to constants and rearrangement, the g_i and h_i are identical. **Hint:** Induct on the total degree of f, and use part (a) to cancel one term at a time.