Math 5385 - Spring 2019
Problem Set 7

Submit solutions to **four** of the following problems.

1. A ring homomorphism is a map $\phi : R \to S$ such that $\phi(0) = 0$, $\phi(1) = 1$, $\phi(a+b) = \phi(a) + \phi(b)$ and $\phi(a \cdot b) = \phi(a) \cdot \phi(b)$. Suppose that ϕ is a surjective ring hom, show that for $I \subset R$ an ideal, the set $\phi(I) \subset S$ is an ideal

2. If $f = a_\ell x^\ell + \cdots + a_0 \in \mathbb{k}[x]$, where $a_\ell \neq 0$ and $\ell > 0$, then the *discriminant* of f is defined to be
$$\operatorname{disc}(f) = \frac{(-1)^{\ell(\ell-1)/2}}{a_\ell} \operatorname{Res}(f, f'; x).$$
   (a) Prove that f has a multiple factor if and only if $\operatorname{disc}(f) = 0$.
   (b) Does $6x^4 - 23x^3 + 32x^2 - 19x + 4$ have a multiple root in $\mathbb{C}$?
   (c) Compute the discriminant of the quadratic polynomial $f = ax^2 + bx + c$. Explain how your answer relates to the quadratic formula.

3. In $\mathbb{Q}[x, y]$, consider $f = x^2y - 3xy^2 + x^2 - 3xy$ and $g = x^3y + x^3 - 4y^2 - 3y + 1$.
   (a) Compute $\operatorname{Res}(f, g; x)$.
   (b) Compute $\operatorname{Res}(f, g; y)$.
   (c) What does the result in part (b) imply about f and g?

4. Consider $f, g \in \mathbb{Q}[x, y]$ and let $J := \langle f, g \rangle \cap \mathbb{Q}[y]$.
   (a) If $f = xy - 1$ and $g = x^2 + y^2 - 4$, then prove that $\operatorname{Res}(f, g; x)$ generates J.
   (b) If $f = xy - 1$ and $g = yx^2 + y^2 - 4$, then prove that $\operatorname{Res}(f, g; x)$ does not generate the ideal J.

5. Suppose that G is a grobner basis for $I \subset k[x_1, \ldots, x_n]$ with respect to the lex order with $x_1 < \cdots < x_n$. Then let $G' = \{g(x_1, a) | g \in G \text{ and } g(x_1, a) \neq 0\}$ the the specialization of G. Then show that
   (a) G' generates the ideal $\{f(x, a) | f \in I\}$
   (b) If $a \in V(I_1)$, then G is a grobner basis for the ideal $\{f(x, a) | f \in I\}$