Math 5385 - Spring 2019
Problem Set 8

Submit solutions to **all** of the following problems.

1. The purpose of this exercise is to show that, if $\mathbb{k}$ is any field which is not algebraically closed, then any affine variety $X \subseteq \mathbb{A}^n(\mathbb{k})$ can be defined by a single equation.
   (a) For a polynomial $f := a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$ of degree m in x, define the *homogenization* to be $f^h := a_m x^m + a_{m-1}x^{m-1}y + \cdots + a_1 x y^{m-1} + a_0 y^m$. Show that f has a root in k if and only if there is $(p, q) \in \mathbb{A}^2(\mathbb{k})$ such that $(p, q) \neq (0, 0)$ and $f^h(p, q) = 0$.
   (b) If $\mathbb{k}$ is not algebraically closed, show that there exists $h \in \mathbb{k}[x, y]$ such that the variety defined by $h = 0$ consists of just the origin.
   (c) If $\mathbb{k}$ is not algebraically closed, show that for each integer $n > 0$ there exists an element $f \in \mathbb{k}[x_1, \ldots, x_n]$ such that the only solution of $f = 0$ is the origin.
   (d) If $X = V(g_1, \ldots, g_r)$ is any affine variety in $\mathbb{A}^n(\mathbb{k})$ where $\mathbb{k}$ is not algebraically closed, then show that X can be defined by a single equation.

2. (a) Show that $\sqrt{\langle x^n, y^m\rangle} = \langle x, y\rangle$
   (b) Find a pair of square free (i.e. not divisible by any square) polynomials such that $\sqrt{\langle f, g\rangle} \neq \langle f, g\rangle$

3. Solve this problem WITHOUT the use of a computer algebra system. (**Hint**: You don't have to use Proposition 4.2.8, although you certainly can)
   Determine whether the following polynomials lie in the given radical ideals. What is the smallest power of the polynomial that lies in the ideal?
   (a) Is $x + y$ in $\sqrt{\langle x^3, y^2, xy(x + y)\rangle}$?
   (b) Is $x^2 + 3xz$ in $\sqrt{\langle x + z, x^2 y, x - z^2\rangle}$?

4. Show the following statements
   (a) $(I_1 + I_2)J = I_1 J + I_2 J$
   (b) $(I_1 \cdots I_r)^m = I_1^m \cdots I_r^m$