# Arithmetic of the moduli of semistable elliptic surfaces


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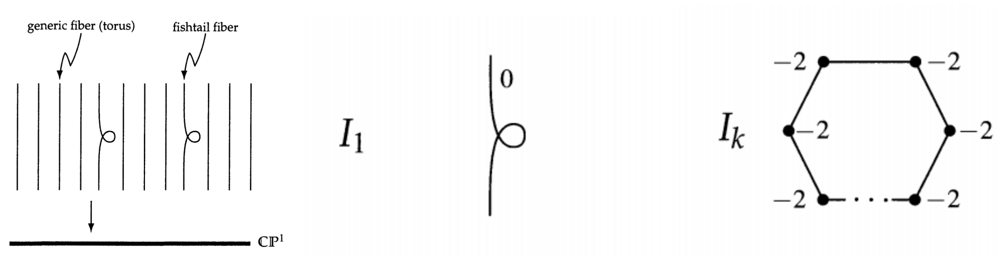
Study of *fibrations* lies at the heart of the Enriques-Kodaira classification of compact complex surfaces as well as the Mumford-Bombieri classification of algebraic surfaces in positive char. The Kodaira dimension $\kappa \in \{-\infty, 0, 1, 2\}$ plays a crucial role in both classifications. In this regard, every elliptic surfaces have $\kappa \leq 1$ and the main classification result for surfaces states that every algebraic surface with $\kappa = 1$ is *elliptic*.

**Semistable elliptic surface** $X$ is a nonsingular surface equipped with a relatively minimal, semistable elliptic fibration $f : X \to \mathbb{P}^1$ that comes with a distinguished section $s : \mathbb{P}^1 \hookrightarrow X$ such that a generic fiber is a smooth curve of genus one and only has at worst nodal singular fibers.

From arithemetic geometry perspective, this can be interpreted as a relative curve over a Dedekind scheme which is the central object in the theory of arithmetic surfaces. Thus, any nonsingular semistable elliptic surface $X$ can be characterized as a family of elliptic curves with squarefree conductor $\mathcal{N}$ as semistable elliptic surfaces contain only nodal singular fibers of fishtail and necklace types $I_k$ $(k \geq 1)$. It has $12n$ nodal singular fibers distributed over $\mu$ distinct singular fibers that are $I_{k_1}, \cdots, I_{k_i}, \cdots, I_{k_\mu}$ with $\sum_{i=1}^{\mu} k_i = 12n = Deg(\Delta(X))$




We formulate the moduli stack $\mathcal{L}_{1,12n}$ of stable elliptic fibrations over $\mathbb{P}^1$ as the Deligne–Mumford algebraic mapping stack of regular morphisms $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$ and establish the bijection between the isomorphism classes of semistable elliptic surfaces and that of stable elliptic fibrations over $\mathbb{P}^1$.

**Moduli stack** $\mathcal{L}_{1,12n}$ of stable elliptic fibrations over $\mathbb{P}^1$ with $12n$ nodal singular fibers and a distinguished section is the Deligne–Mumford mapping stack for any field $K$ of characteristic $\neq 2, 3$

$$\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}) \cong \mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(4,6))$$

.

$$\begin{array}{ccccc} X & \xrightarrow{\nu} & Y = \varphi_f^*(\overline{\mathcal{C}}_{1,1}) & \longrightarrow & \overline{\mathcal{C}}_{1,1} \\ \downarrow f & & \downarrow g & & \downarrow p \\ \mathbb{P}^1 & = & \mathbb{P}^1 & \xrightarrow{\varphi_f} & \overline{\mathcal{M}}_{1,1} \end{array}$$

Note that the stable reduction $\nu$ introduces singularities of type $A_k$ on $Y$. Conversely, given any regular morphism $\varphi_g \in \mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$, the total space $Y = \varphi_f^*(\overline{\mathcal{C}}_{1,1})$ of the stable family $g : \varphi_f^*(\overline{\mathcal{C}}_{1,1}) \to \mathbb{P}^1$ can have $A_k$ singularities which corresponds to the regular morphism $\varphi_g \in \mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$ being ramified over the nodal divisor point $[\infty] = \overline{\mathcal{M}}_{1,1} \smallsetminus \mathcal{M}_{1,1}$ of order $k-1$.

To understand $\mathcal{L}_{1,12n}$, the regular morphism $\varphi_f$ from $\mathbb{P}^1 \to \mathcal{P}(4,6)$ is equivalent to considering the line bundles on $\mathbb{P}^1$ which are $\mathcal{L} \simeq \varphi_f^* \mathcal{O}_{\mathcal{P}(4,6)}(1)$ of degree $n$ together with the global sections $(u, v) \in \mathcal{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(4n)) \oplus H^0(\mathcal{O}_{\mathbb{P}^1}(6n)))$. Fixing an affine chart $\mathbb{A}^1_t \hookrightarrow \mathbb{P}^1$, those sections become polynomials of variable $t$. By taking a look at all possible degrees of $u$ and $v$ gives the following stratification:

$$L_{1,12n} = F_{4n,6n} \sqcup \left( \bigsqcup_{k=0}^{4n-1} F_{k,6n} \right) \sqcup \left( \bigsqcup_{l=0}^{6n-1} F_{4n,l} \right)$$

$$L_{1,12n} = \overline{F_{4n,6n}} \supsetneq \overline{F_{4n-1,6n}} \supsetneq \cdots \supsetneq \overline{F_{0,6n}} = F_{0,6n}$$

$$L_{1,12n} = \overline{F_{4n,6n}} \supsetneq \overline{F_{4n,6n-1}} \supsetneq \cdots \supsetneq \overline{F_{4n,0}} = F_{4n,0}$$

$$\overline{F_{4n-k,6n}} \cap \overline{F_{4n,6n-l}} = \emptyset \;\forall k, l > 0$$

Where $F_{d_1,d_2}$ parametrizes such pairs $(u, v)$ with $\deg(u) = d_1$ and $\deg(v) = d_2$ with respect to $t$. This can be understood by generalizing [3].

To count $\mathbb{F}_q$-points on $L_{1,12n}(\mathbb{F}_q)$, we will use the idea of cut-and-paste by Grothendieck. Fix a field $K$.

**Grothendieck ring $K_0(\mathrm{Var}_K)$ of $K$-varieties** is a group generated by isomorphism classes of $K$-varieties $[X]$, modulo *scissor* relations $[X] = [Z] + [X - Z]$ for $Z \subset X$ a closed subvariety. Multiplication on $K_0(\mathrm{Var}_K)$ is induced by $[X][Y] := [X \times_K Y]$. There is a distinguished element $\mathbb{L} := [\mathbb{A}^1] \in K_0(\mathrm{Var}_K)$, called the *Lefschetz motive.*

We can express $[L_{1,12n}]$ as linear combination of classes of other varieties.

$$[L_{1,12n}] = [F_{4n,6n}] + \sum_{k=0}^{4n-1} [F_{k,6n}] + \sum_{l=0}^{6n-1} [F_{4n,l}]$$

**Motive count of the moduli** $L_{1,12n}$ is the class $[L_{1,12n}]$ in $K_0(\mathrm{Var}_K)$ with $\mathrm{char}(K) \neq 2, 3$ for $L_{1,12n}$ the coarse moduli space for semistable elliptic fibrations over $\mathbb{P}^1$ is equivalent to

$$[L_{1,12n}] = \mathbb{L}^{10n+1} - \mathbb{L}^{10n-1}$$

It is easy to see that for $K = \mathbb{F}_q$, the assignment $[X] \mapsto |X(\mathbb{F}_q)|$ gives a well-defined ring homomorphism $\#_q : K_0(\mathrm{Var}_{\mathbb{F}_q}) \to \mathbb{Z}$, which allows us to deduce $|L_{1,12n}(\mathbb{F}_q)|$.

**Point count of the moduli** $L_{1,12n}$: If $q = p^k$ and $2, 3 \nmid q$, then

$$|L_{1,12n}(\mathbb{F}_q)| = q^{10n+1} - q^{10n-1}$$

Counting points of $\mathcal{L}_{1,12n}$ is the same as counting points of moduli of nonsingular semistable elliptic surfaces.

Lastly, we consider the unifying principle of twentieth-century number theory called the *global fields analogy* which is the observation that for any finite field $\mathbb{F}_q$, the finite extensions of the function field $\mathbb{F}_q(t)$ have much in common with the finite extensions of $\mathbb{Q}$. Let $K$ be a global field and $\mathcal{O}_K$ be its ring of integers such as

1. The function field $K = \mathbb{F}_q(t)$ with $\mathcal{O}_K = \mathbb{F}_q[t]$
2. The number field $K = \mathbb{Q}$ with $\mathcal{O}_K = \mathbb{Z}$

We define the height of a maximal ideal $\mathfrak{p} \in Spec\ \mathcal{O}_K$ to be $ht(\mathfrak{p}) := |\mathcal{O}_K/\mathfrak{p}|$ the cardinality of the residue field $\mathcal{O}_K/\mathfrak{p}$. As the discriminant divisor $\Delta(X)$ is an invariant of the choice of model $f : X \to \mathbb{P}^1$, we count the number of isomorphism classes of semistable elliptic fibrations on $\mathbb{F}_q(t)$ by the bounded height of $\Delta(X)$

$$ht(\Delta(X)) = \prod_{i=1}^{\mu} |\mathbb{F}_q|^{k_i} = q^{k_1} \cdots q^{k_i} \cdots q^{k_\mu} = q^{k_1 + \cdots + k_\mu} = q^{12n}$$

Define a function $Z_K(B)$ which in the arithmetic function field case is $Z_{\mathbb{F}_q(t)}(B)$.

**Counting of semistable elliptic fibrations** is $Z_{\mathbb{F}_q(t)}(B) :=$
{ # Nonsingular semistable elliptic fibrations over $\mathbb{P}^1_{\mathbb{F}_q}$ with $12n$ nodal singular fibers and a distinguished section counted by $0 < ht(\Delta(X)) = \prod_{\mathfrak{p}} ht(\mathfrak{p}) = q^{12n} \leq B$ }

We now compute the $Z_{\mathbb{F}_q(t)}(B)$ by the arithmetic invariant $|L_{1,12n}(\mathbb{F}_q)|$.

**Computation of $Z_{\mathbb{F}_q(t)}(B)$ :** The growth rate of $Z_{\mathbb{F}_q(t)}(B)$ is a polynomial of degree $\frac{5}{6}$ in $B$.

$$Z_{\mathbb{F}_q(t)}(B) = \sum_{n=1}^{\frac{\log_q B}{12}} |L_{1,12n}(\mathbb{F}_q)| \leq \frac{(q^{11} - q^9)}{(q^{10} - 1)} \cdot (B^{\frac{5}{6}} - 1) \sim \mathcal{O}(B^{\frac{5}{6}})$$

Switching to the number field realm with $K = \mathbb{Q}$ and $\mathcal{O}_K = \mathbb{Z}$, in order to match with the function field, we define $ht(\Delta)$ to be the cardinality of the ring of functions on the subscheme $Spec(\mathbb{Z}/(\Delta))$. This leads to the following analog of $Z_K(B)$ on the number field $\mathbb{Q}$ which is $Z_{\mathbb{Q}}(B)$.

**Heuristic for counting semistable elliptic curves over $\mathbb{Q}$ :** Through the global fields analogy we conjecture that asymptotic growth rate of $Z_{\mathbb{Q}}(B)$, the counting of semistable elliptic curves over $\mathbb{Q}$ by $ht(\Delta) \leq B$, follows from the polynomial growth rate of $Z_{\mathbb{F}_q(t)}(B) \sim \mathcal{O}(B^{\frac{5}{6}})$

The semistable elliptic curves were used by Andrew Wiles to prove *Taniyama-Shimura-Weil conjecture* now known as *modularity theorem* which was enough to establish *Fermat's last theorem* as a true theorem.

Continuing from the semistable elliptic fibrations moduli, we consider the following directions.

- Motive/Point conting on the *compact moduli* by giving orbifold structure on the domain $\mathbb{P}^1$ and count twisted stable maps such as $\overline{\mathcal{L}}_{1,12n} = \mathrm{Hom}_n[\mathcal{P}(a, b), \mathcal{P}(4, 6)]$ which will lead to new heuristics with unstable singular fibers in elliptic fibrations. The choice of $a$ and $b$ give constraints on which types of unstable fibers occur.
- *Topology of the moduli* - Étale cohomology and weights of the interior of the moduli. Focus is on showing interesting relationship between compactly supported cohomology of interior, cohomology of compactification (KSBA through twisted stable maps), and its automorphic forms.
- *Mordell-Weil group* MW(X) of an elliptic surface $X$ in relation to the moduli $\mathcal{L}_{1,12n} = \mathrm{Hom}_n[\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}]$ and its arithmetic invariant $[L_{1,12n}] = \mathbb{L}^{10n+1} - \mathbb{L}^{10n-1}$ through the *Tate-Shioda formula.*
- For genus 2 fibrations moduli, we similarly consider $\mathcal{L}_{2,10m} = \mathrm{Hom}_n[\mathbb{P}^1, \overline{\mathcal{M}}_2]$ and reach the following point count $|L_{2,10m}(\mathbb{F}_q)| \sim \mathcal{O}(q^{22m+3})$ via minimal model $\overline{\mathcal{M}}_2 \to \mathbb{P}^6 /\!/ SL_2 \cong \mathcal{P}(2, 4, 6, 10)$ and consider $\mathrm{Hom}_m[\mathbb{P}^1, \mathcal{P}(2, 4, 6, 10)]$.
- A future goal is to study $\mathrm{Hom}_n[\mathbb{P}^1, \overline{\mathcal{M}}_{K3}]$ which would be the *Moduli of K3 fibered threefolds.* Currently, it is not known what $\overline{\mathcal{M}}_{K3}$ is, but for special types of $K3$'s such compactification exists.

**References**


1. Changho Han and Jun–Yong Park *Arithmetic of the moduli of semistable elliptic surfaces,* arXiv:1607.03187. submitted.
2. Kenneth Ascher, Dori Bejleri and Jun-Yong Park *Moduli of fibered surfaces and heuristics for counting hyperelliptic curves over global fields,* work in progress.
3. Benson Farb and Jesse Wolfson *Topology and arithmetic of resultants, I: spaces of rational maps,* New York Journal of Mathematics, **22**, (2016): 801–821.



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