Math 126 Final Exam

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Name:________________________________

Instructions:

1. I recommend looking over the problems first and starting with those you feel most comfortable with.
2. Unless otherwise noted, be sure to include explanations to justify each step in your arguments and computations. For example, make sure to check that the hypotheses of any theorems you use are satisfied.
3. All work should be done in the space provided in this exam booklet. Cross out any work you do not wish to be considered. Additional white paper is available if needed.
4. Books, notes, calculators, cell phones, pagers, or other similar devices are not allowed during the exam. Please turn off cell phones for the duration of the exam.
5. If you complete the exam within the last 15 minutes, please remain in your seat until the examination period is over.
6. In the event that it is necessary to leave the room during the exam (e.g., fire alarm), this exam and all your work must remain in the room, face down on your desk.

Problem	Score
1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
7	/10
8	/10
9	/10
10	/10
Total:	/100

1. (a) Give an example of a linear homogeneous PDE, a linear inhomogeneous PDE, and a nonlinear PDE. [3 points]

   (b) State in words what it means for a PDE to be well-posed. [4 points]

   (c) Give an example of a PDE that is not well-posed. [3 point]

2. Find the solution $u(x, y)$ of the linear partial differential equation $u_x + yu_y = 0$ that satisfies $u(0, y) = \sin(y)$. [10 points]

3. (a) Write down d'Alembert's formula for the solution of the one dimensional wave equation $u_{tt} = c^2 u_{xx}$ with initial conditions $u(x, 0) = \varphi(x)$ and $u_t(x, 0) = \psi(x)$. [4 points]

(b) Solve the wave equation $u_{tt} = c^2 u_{xx}$ with initial conditions $u(x, 0) = x^2$ and $u_t(x, 0) = \cos(x)$. [6 points]

4. Find the solution $u(x,t)$ of the heat equation $u_t = u_{xx}$ on the halfline $x > 0$ satisfying $u(x,0) = 1$ for $x > 0$ and $u(0,t) = 0$ for $t > 0$. [10 points]

5. Suppose that

$$-\Delta u + u = f$$

within a bounded open set $D \subseteq \mathbb{R}^3$, and $u = 0$ on the boundary ∂D. Show that

$$\iiint_D |\nabla u|^2 \, dxdydz \leq \frac{1}{2} \iiint_D f^2 \, dxdydz.$$

[10 points] [Hint: Multiply both sides of the PDE by u and integrate over D. Then apply Green's identity and make use of the inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$.]

6. Suppose that

$$\Delta u = 0$$

within a bounded open set $D \subseteq \mathbb{R}^2$. Show that

$$\max_{D} |\nabla u| = \max_{\partial D} |\nabla u|,$$

that is, the length of ∇u attains its maximum on ∂D. [10 points] [Hint: Set $v = |\nabla u|^2 = u_x^2 + u_y^2$, compute Δv, and use the maximum principle.]

7. A weak solution of Burger's equation

$$u_t + uu_x = 0 \quad \text{for } x \in \mathbb{R},\ t > 0$$

has the form

$$u(x,t) = \begin{cases} \frac{x}{t+1}, & \text{for } x < s(t) \\ 0, & \text{for } x > s(t), \end{cases}$$

where $s(t)$ is a shock curve starting at $s(0) = 1$. Find a formula for $s(t)$. [10 points] [Hint: Use the Rankine-Hugoniot condition to find an ODE that $s(t)$ satisfies.]

8. Consider the following implicit scheme for the heat equation

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2}.$$

Let $s = \frac{\Delta t}{\Delta x^2}$. Prove that the scheme is unconditionally stable (that is, it is stable for all values of s). [10 points]

9. Let f be a distribution and ψ be an infinitely differentiable function. We define the product ψf to be the distribution

$$(\psi f, \varphi) := (f, \psi\varphi) \quad \text{for all test functions } \varphi.$$

(a) Show that $e^x\delta(x) = \delta(x)$ as distributions, where δ is the Delta function. [3 points]

(b) Let f be a distribution and ψ an infinitely differentiable function. Show that the product rule

$$(\psi f)' = \psi f' + \psi' f$$

holds in the distributional sense. [7 points]

10. Let $\{\alpha_n\}_{n=1}^{\infty}$ denote the positive solutions of the equation $\alpha \tan(\alpha) = 1$. Find the solution $u(x, y)$ of the boundary-value problem

$$\begin{cases} \Delta u = 0, & 0 < x < 1, \quad y > 0 \\ u_x(0, y) = 0, & y > 0 \\ u(1, y) + u_x(1, y) = 0, & y > 0 \\ u(x, 0) = 1, & 0 < x < 1 \end{cases}$$

that is bounded as $y \to \infty$. [10 points] [Hint: Use separation of variables and look for a series solution. You do not need to prove convergence of the series.]

Scratch paper