MATH 126 – HOMEWORK 12 (DUE FRIDAY DEC 11)

1. (a) Show that the Fourier Transform of $e^{-a|x|}$ is $\frac{2a}{a^2+k^2}$.

   (b) Show that the Fourier Transform of the square pulse $H(a-|x|)$ is $\frac{2\sin(ak)}{k}$, where H is the Heaviside function.

   (c) Show that the Fourier Transform of $xf(x)$ is $i\frac{dF}{dk}$. [You may assume f is continuously differentiable and vanishes as $x \to \pm\infty$.]

2. (a) Show that $(f * g)' = f' * g = f * g'$.

   (b) Show that $f * (g * h) = (f * g) * h$.

3. **Nyquist-Shannon sampling theorem**: Let f be an integrable function whose Fourier transform F has compact support in $[-\Omega, \Omega]$[1]. Fix $\ell \geq \Omega$.

   (a) Show that

$$F(k) = \frac{\pi}{\ell} \sum_{n=-\infty}^{\infty} f\left(\tfrac{n\pi}{\ell}\right) e^{-in\pi k/\ell} \quad \text{for } -\ell \leq k \leq \ell. \tag{1}$$

   [Hint: Write down the complex form of the Fouier series for F on the interval $(-\ell, \ell)$. Simplify the coefficients with the inverse Fourier transform formula. You may assume that the Fourier series converges.]

   (b) Show that

$$f(x) = \sum_{n=-\infty}^{\infty} f\left(\tfrac{n\pi}{\ell}\right) \operatorname{sinc}(\ell x - n\pi) \quad \text{for all } x \in \mathbb{R}, \tag{2}$$

   where $\operatorname{sinc}(x) := \sin(x)/x$ for $x \neq 0$, and $\operatorname{sinc}(0) := 1$. [Hint: Express f via the inverse Fourier transform formula and substitute the expression (1) for F. You may exchange the infinite summation and integral without justification.]

   (c) Part (b) shows that f can be exactly reconstructed from evenly spaced samples $f\left(\frac{n\pi}{\ell}\right)$ for $n \in \mathbb{Z}$ provided $\ell \geq \Omega$. Find an expression for the *sampling frequency* f_s in terms of ℓ (use units of # samples/unit length). The *bandwidth* of f, denoted f_b, is the highest frequency present in the Fourier representation

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) e^{ikx}\, dk.$$

   Due to the compact support of F, f_b is the frequency of the complex wave

$$e^{\pm i\Omega x} = \cos(\Omega x) \pm i \sin(\Omega x).$$

   Find an expression for f_b, and show that $\ell \geq \Omega$ is equivalent to $f_s \geq 2f_b$. [This is the famous Nyquist-Shannon sampling theorem: A signal can be exactly reconstructed from evenly spaced samples provided the sampling frequency is at least twice the bandwidth of the signal. The reconstruction formula is given in (2), and is often called *sinc interpolation*.]

[1] This means $F(k) = 0$ for $|k| \geq \Omega$.

4. Use Fourier Transforms to solve the ODE $-u_{xx} + a^2 u = \delta$, where δ is the Delta function.

5. Use the Fourier Transform method to find a representation formula for the solution $u(x,t)$ of the inhomogeneous heat equation with zeroth order term

$$\left.\begin{aligned} u_t - u_{xx} + cu = f, &\quad \mathbb{R} \times \{t > 0\} \\ u = \varphi, &\quad \mathbb{R} \times \{t = 0\}, \end{aligned}\right\}$$

where $f = f(x,t)$ and $c \in \mathbb{R}$. [Hint: Let $\hat{u}(k,t)$ denote the Fourier Transform of u in the variable x, let F denote the Fourier Transform of f and let Φ denote the Fourier Transform of φ. Apply Fourier Transforms to both sides of the PDE, and the intial condition, to find an ODE for $\hat{u}(k,t)$ in the variable t. Solve the ODE and then invert the Fourier Transform using the convolution property of the Fouier Transform. Your solution should agree with Duhamel's principle from HW 2 when $c = 0$.]

6. Solve the inviscid Burger's equation $u_t + uu_x = 0$ with initial condition $u(x,0) = x$. Sketch some of the characteristic curves.

7. Solve $u_t + u^2 u_x = 0$ with $u(x,0) = 2 + x$. Sketch some of the characteristics.