Math 126 – Homework 3 (Due Monday Sept 21)

1. Show that the wave equation does not, in general, satisfy a maximum principle.

2. For a solution $u(x,t)$ of the wave equation

$$u_{tt} - u_{xx} = 0,$$

the energy density is defined as $e = (u_t^2 + u_x^2)/2$ and the momentum density is $p = u_t u_x$.

   (a) Show that $e_t = p_x$ and $p_t = e_x$.

   (b) Show that both e and p also satisfy the wave equation.

3. Consider a traveling wave $u(x,t) = f(x - at)$, where f is a given function of one variable.

   (a) Show that if u is a solution of the wave equation $u_{tt} - c^2 u_{xx} = 0$, then $a = \pm c$ (unless f is a linear function).

   (b) Show that if u is a solution of the diffusion equation $u_t - k u_{xx} = 0$ then

$$u(x,t) = C_1 \exp\Big(-\frac{a}{k}(x - at)\Big) + C_2,$$

   where C_1 and C_2 are arbitrary constants, and $a \in \mathbb{R}$ is arbitrary.

This exercise shows that the speed of propagation of travelling wave solutions of the wave equation is c, while for the heat equation, any arbitrary speed $a \in \mathbb{R}$ will do (i.e., we have infinite speed of propagation).

4. Here, we find a direct relationship between the heat and wave equations. Let $u(x,t)$ solve the wave equation on the whole line, and suppose the second derivatives of u are bounded. Let

$$v(x,t) = \frac{c}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{-s^2c^2/4kt} u(x,s)\, ds.$$

   (a) Show that $v(x,t)$ solves the diffusion equation $u_t - k u_{xx} = 0$.

   (b) Show that $\lim_{t\to 0} v(x,t) = u(x,0)$.

5. Find a formula for the solution of

$$\left.\begin{aligned} u_t - k u_{xx} &= 0 \quad \text{for } x > 0 \text{ and } t > 0 \\ u &= 0 \quad \text{for } x > 0 \text{ and } t = 0 \\ u &= 1 \quad \text{for } x = 0 \text{ and } t > 0. \end{aligned}\right\} \tag{1}$$

6. Suppose u solves the heat equation

$$\left.\begin{aligned} u_t - k u_{xx} &= 0 \quad \text{for } x \in \mathbb{R} \text{ and } t > 0 \\ u &= \varphi \quad \text{for } x \in \mathbb{R} \text{ and } t = 0. \end{aligned}\right\} \tag{2}$$

Show that if φ is odd, then for each $t > 0$ the function $x \mapsto u(x,t)$ is odd. Show that the analogous result is true when φ is even.

7. Consider the diffusion equation on the half-line with Robin boundary condition:

$$\left.\begin{aligned} u_t - ku_{xx} &= 0 \quad \text{for } x > 0 \text{ and } t > 0 \\ u &= \varphi \quad \text{for } x > 0 \text{ and } t = 0 \\ u_x - hu &= 0 \quad \text{for } x = 0 \text{ and } t > 0. \end{aligned}\right\} \tag{3}$$

(a) Motivated by the method of odd and even extensions, we might guess that the solution $u(x,t)$ is of the form

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y)\, dy, \tag{4}$$

where

$$f(y) = \begin{cases} \varphi(y), & \text{if } y > 0 \\ g(-y), & \text{if } y \leq 0, \end{cases} \tag{5}$$

and g is some (yet to be determined) function satisfying $g(0) = \varphi(0)$. Recall that for the method of odd extension, $g = -\varphi$, and for the method of even extension $g = \varphi$. Show that u given by (4) is the solution of (3) provided the function $f' - hf$ is *odd*.

(b) Show that $f' - hf$ is odd if and only if

$$g'(y) + hg(y) = \varphi'(y) - h\varphi(y) \text{ for all } y \geq 0.$$

(c) Suppose that $\varphi(y) = y^2$. Find g by solving the ODE from part (b) with initial condition $g(0) = \varphi(0)$ assuming $h \neq 0$. What happens when $h = 0$?