Math 126 – Homework 4 (Due Friday Sept 25)

1. **Maximum principle:** Consider the heat equation

$$\text{(H)} \quad \begin{cases} u_t - u_{xx} = 0, & -\infty < x < \infty,\ t > 0 \\ u(x,0) = \varphi(x), & -\infty < x < \infty. \end{cases}$$

As it turns out, there are infinitely many solutions u of the above heat equation. All but one solution are "non-physical" and grow exponentially fast as $x \to \pm\infty$ (see, e.g., Fritz John, Partial Differential Equations, Chapter 7). In this question, you will show that if $\varphi \leq M$, then any bounded solution u of (H) satisfies $u \leq M$. This is a maximum principle for (H), and can be used to establish uniqueness of bounded solutions of (H).

Throughout the question let u be a bounded solution of (H); this means there exists $C > 0$ such that $|u(x,t)| \leq C$ for all (x,t).

(a) Show that $w(x,t) = x^2 + 2t$ solves the heat equation.

(b) For every $\varepsilon > 0$ show that

$$u(x,t) \leq \varepsilon w(x,t) + M \quad \text{for all } x \in \mathbb{R} \text{ and } t > 0,$$

where $M > 0$ is any number satisfying $\varphi(x) \leq M$ for all $x \in \mathbb{R}$. [Hint: For $N > 0$ let R_N denote the rectangle

$$R_N = [-N, N] \times [0, N].$$

Show that there exists $\overline{N} > 0$ such that for all $N > \overline{N}$, $u \leq \varepsilon w + M$ on the sides and base of R_N. Then apply the comparison principle from HW2, problem 7.]

(c) Let $M > 0$ such that $\varphi(x) \leq M$ for all $x \in \mathbb{R}$. Show that $u \leq M$.

(d) Show that there is at most one bounded solution u of (H) when φ is bounded. [Hint: Take two bounded solutions u, v and consider $w = u - v$.]

It is possible to prove a stronger result; namely that there is at most one solution u of (H) satisfying the exponential growth estimate

$$u(x,t) \leq Ce^{x^2}$$

for a constant $C > 0$. The proof is similar to this exercise, except that w has a different form (given in HW2 #6). This means that the "non-physical" solutions all grow faster than e^{x^2} as $x \to \pm\infty$.

2. (a) Use the Fourier expansion to explain why the note produced by a violin string rises by one octave when the string is clamped exactly at its midpoint. [Each increase by an octave corresponds to a doubling of the frequency.]

(b) Explain why the pitch of the note rises when the string is tightened.

3. A quantum-mechanical particle on the line with an infinite potential outside the interval $(0, l)$ ("particle in a box") is given by Schrödinger's equation

$$u_t = iu_{xx} \quad \text{on } (0, l),$$

with Dirichlet conditions at the ends. Separate variables to find its representation as a series.

4. Consider waves in a resistant medium, which satisfy the equation

$$\text{(W)} \quad \begin{cases} u_{tt} - c^2 u_{xx} + r u_t = 0, & 0 < x < l,\ t > 0 \\ u(x,0) = \varphi(x), & 0 < x < l \\ u_t(x,0) = \psi(x), & 0 < x < l \\ u(0,t) = u(l,t) = 0 & t > 0, \end{cases}$$

where r is a constant, $0 < r < 2\pi c/l$. Write down the series expansion of the solution.

5. Consider the equation $u_{tt} = c^2 u_{xx}$ for $0 < x < l$ with boundary conditions $u_x(0,t) = u(l,t) = 0$.

   (a) Show that the eigenfunctions are $X_n(x) = \cos\big((n + \frac{1}{2})\pi x/l\big)$.

   (b) Write down the series expansion for a solution $u(x,t)$.

6. Consider diffusion inside an enclosed circular tube. Let its length (circumference) be $2l$. Let x denote the arc length parameter where $-l < x < l$. Then the concentration of the diffusion substance satisfifes

$$\begin{cases} u_t - k u_{xx} = 0, & -l < x < l,\ t > 0 \\ u(-l,t) = u(l,t) & t > 0 \\ u_x(-l,t) = u_x(l,t) & t > 0. \end{cases}$$

These are called *periodic boundary conditions.*

   (a) Show that the eigenvalues are $\lambda = (n\pi/l)^2$ for $n = 0, 1, 2, 3, \ldots$.

   (b) Show that the concentration is

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{\pi n x}{l}\right) + B_n \sin\left(\frac{\pi n x}{l}\right) \right) \exp\left(\frac{-n^2\pi^2 k t}{l^2}\right).$$

7. Let $\varphi(x) = x^2$ for $0 \le x \le 1 = l$.

   (a) Calculate the Fourier sine series for φ.

   (b) Calculate the Fourier cosine series for φ.

8. Find the Fourier cosine series of the function $|\sin(x)|$ in the interval $(-\pi, \pi)$. Use it to find the sums

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1}.$$