MATH 126 – HOMEWORK 5 (DUE FRIDAY OCT 2)

1. For each of the following functions, state whether it is even, odd, or neither, and whether it is periodic. If periodic, what is the smallest period?
   (a) $\sin(ax)$ for $a > 0$
   (b) e^{ax} for $a > 0$
   (c) x^m for an integer m
   (d) $\tan(x^2)$
   (e) $|\sin(x/b)|$ for $b > 0$
   (f) $x\cos(ax)$ for $a > 0$
2. Show that $\cos(x) + \cos(ax)$ is periodic if a is a rational number. What is its period?
3. Recall a function f is Lipschitz if there exists $L > 0$ such that
   $$|f(x) - f(y)| \leq L|x - y| \quad \text{for all } x, y$$
   (a) Show that every Lipschitz function f is continuous. [Hint: A function is continuous if $\lim_{y\to x} f(y) = f(x)$ for all x.]
   (b) Show that if f is continuously differentiable and f' is bounded, then f is Lipschitz.
4. Show that the Fourier sine series on $(0, l)$ can be derived from the full Fourier series on $(-l, l)$ by using the odd extension of the function.
5. Show that any function f on the real line $\mathbb{R}$ can be written as the sum of an even and odd function, i.e. $f = \varphi + \psi$ where φ is even and ψ is odd.
6. Consider the geometric series $\sum_{n=0}^{\infty}(-1)^n x^{2n}$.
   (a) Does it converge pointwise in the interval $-1 < x < 1$?
   (b) Does it converge uniformly in the interval $-1 < x < 1$?
   (c) Does it converge in the L^2 sense in the interval $-1 < x < 1$? [Hint: You can compute the partial sums explicitly.]
7. Prove the Cauchy-Schwarz inequality
   $$|\langle f, g\rangle| \leq \|f\|\|g\|,$$
   for any pair of functions f and g on an interval (a, b). [Hint: Consider the expression $h(t) := \|f + tg\|^2$ where $t \in \mathbb{R}$, and find the value of t that minimizes h.]
8. Prove the Cauchy-Schwarz inequality for infinite series
   $$\sum_{n=1}^{\infty} a_n b_n \leq \left(\sum_{n=1}^{\infty} a_n^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} b_n^2\right)^{\frac{1}{2}}.$$
   [Hint: Use an argument similar to that for the previous question. Prove it first for finite sums, and then pass to the limit.]

9. Show that if f is a continuously differentiable 2π-periodic function satisfying

$$\int_{-\pi}^{\pi} f(x)\,dx = 0,$$

then we have

$$\int_{-\pi}^{\pi} f(x)^2\,dx \leq \int_{-\pi}^{\pi} f'(x)^2\,dx.$$

[Hint: Use Parseval's identity. Also use integration by parts to find a relationship between the Fourier coefficients of f and f'.]