MATH 126 – HOMEWORK 6 (DUE FRIDAY OCT 16)

1. Consider the wave equation

$$\text{(W)} \quad \begin{cases} u_{tt} - u_{xx} = 0, & 0 < x < \pi,\ t > 0 \\ u(0,t) = u(\pi,t) = 0, & t > 0 \\ u(x,0) = \varphi(x), & 0 < x < \pi \\ u_t(x,0) = \psi(x), & 0 < x < \pi. \end{cases}$$

   (a) Write down the Fourier series representation of the solution $u(x,t)$, including the formulas for the coefficients.

   (b) Explain why we cannot conclude that $u(x,t)$ is infinitely differentiable, as we did for the heat equation in class.

   (c) Suppose that $\varphi \in C^4_{per}$ and $\psi \in C^3_{per}$. Use the results from class about Fourier series regularity to show that $x \mapsto u(x,t)$ is an element of C^2_{per} for every $t > 0$ and find an expression for u_{xx}.

   (d) As we did for the heat equation in class, use the dominated convergence theorem to show that u_t and u_{tt} exist, and write down and expression for u_{tt}.

   (e) Show that your formula from (a) actually solves the wave equation (W).

2. Define

$$f(x) = \begin{cases} x(\pi - x), & 0 < x < \pi \\ x(\pi + x), & -\pi < x \leq 0. \end{cases}$$

   Let A_n and B_n denote the Fourier coefficients of f on the interval $(-\pi, \pi)$. Since f is an odd function, $A_n = 0$ for all n.

   (a) Find all integers k for which $f \in C^k_{per}$ (here you can identify f with its 2π-periodic extension).

   (b) Without computing B_n, explain how you know that there exists a constant $C > 0$ such that

$$|B_n| \leq \frac{C}{n}.$$

   (c) Without computing B_n, explain why it is impossible that $B_n = \frac{1}{n^4}$.

3. Solve $u_{xx} + u_{yy} + u_{zz} = 0$ in the spherical shell $0 < a < r < b$ with boundary conditions $u = A$ on $r = a$, and $u = B$ on $r = b$, where A and B are constants and $r = \sqrt{x^2 + y^2 + z^2}$. [Hint: Look for a solution depending only on r.]

4. A function u is *subharmonic* in D if $-\Delta u \leq 0$ in D. Let $u(x,y)$ be subharmonic in an open and bounded set $D \subseteq \mathbb{R}^2$. Modify the proof of the mean value property from class to show that

$$u(x,y) \leq \frac{1}{2\pi} \int_0^{2\pi} u(x + r\cos\theta, y + r\sin\theta)\, d\theta,$$

for all $r > 0$ such that the ball of radius r centered at (x, y) belongs to D. Integrate the expression above in polar coordinates to show that

$$u(\mathbf{x}_0) \leq \frac{1}{\pi r^2} \iint_{B(\mathbf{x}_0, r)} u(\mathbf{x})\, d\mathbf{x},$$

whenver the ball $B(\mathbf{x}_0, r)$ is contained in D. Here, $\mathbf{x} = (x, y)$ and $\mathbf{x}_0 = (x_0, y_0)$ denote points in $\mathbb{R}^2$, and $d\mathbf{x} = dxdy$.

5. Let $u(x, y)$ be subharmonic in a bounded, open, and connected set $D \subseteq \mathbb{R}^2$. Show that if u attains its maximum value over D at a point $\mathbf{x} = (x, y) \in D$, then u is constant in D.