MATH 126 – HOMEWORK 8 (DUE FRIDAY OCT 30)

1. Let $G(\mathbf{x}, \mathbf{x}_0)$ be the Green's function for an open and bounded set D.

   (a) Let u be a solution of
   $$\begin{cases} \Delta u = f & \text{in } D \\ u = 0 & \text{on } \partial D. \end{cases}$$
   Show that
   $$u(\mathbf{x}_0) = \iiint_D f(\mathbf{x}) G(\mathbf{x}, \mathbf{x}_0)\, d\mathbf{x}. \tag{1}$$
   [Hint: Denote the right hand side of (1) by A. Let $D_\varepsilon = D \setminus B(\mathbf{x_0}, \varepsilon)$ and write
   $$A = \iiint_{D_\varepsilon} \Delta u(\mathbf{x}) G(\mathbf{x}, \mathbf{x}_0)\, d\mathbf{x} + \int_{B(\mathbf{x_0}, \varepsilon)} f(\mathbf{x}) G(\mathbf{x}, \mathbf{x}_0)\, d\mathbf{x}.$$
   Show that the second term converges to zero as $\varepsilon \to 0$. Use Green's second identity to show that the first term converges to $u(\mathbf{x}_0)$ as $\varepsilon \to 0$.]

   (b) Use part (a) and results from class to conclude that the solution u of
   $$\begin{cases} \Delta u = f & \text{in } D \\ u = g & \text{on } \partial D \end{cases}$$
   is given by
   $$u(\mathbf{x}_0) = \iint_{\partial D} g(\mathbf{x}) \frac{\partial G}{\partial \hat{\mathbf{n}}}(\mathbf{x}, \mathbf{x}_0)\, dS(\mathbf{x}) + \iiint_D f(\mathbf{x}) G(\mathbf{x}, \mathbf{x}_0)\, d\mathbf{x}.$$

2. Show that the Green's function is unique. [Hint: Take the difference of two of them and use the maximum principle or energy methods.]

3. Find the one-dimensional Green's function for the interval $(0, \ell)$. The three properties defining it can be restated as follows

   (a) It solves $G''(x) = 0$ for $x \neq x_0$

   (b) $G(0) = G(\ell) = 0$

   (c) $G(x)$ is continuous at x_0 and $G(x) + \frac{1}{2}|x - x_0|$ is harmonic at x_0.

4. Find the Green's function for the tilted half-space
   $$D = \{(x, y, z) \,:\, ax + by + cz > 0\}.$$
   [Hint: Use the Green's function for the halfspace $\{z > 0\}$ and a change of variables.]

5. Find the Green's function for the half ball
   $$D = \{(x, y, z) \,:\, x^2 + y^2 + z^2 < a^2 \ \text{ and } \ z > 0\}.$$
   [Hint: Reflect the solution for the whole ball across the plane $z = 0$.]

6. Consider the two dimensional disk

$$D = \{(x, y) \,:\, x^2 + y^2 < a^2\}.$$

Show that the Green's function for the disk is

$$G(\mathbf{x}, \mathbf{x}_0) = \frac{1}{2\pi}\log(|\mathbf{x} - \mathbf{x}_0|) - \frac{1}{2\pi}\log\left(\frac{|\mathbf{x}_0|}{a}|\mathbf{x} - \mathbf{x}_0^*|\right).$$

where $\mathbf{x}_0^* = \frac{a^2\mathbf{x}_0}{|\mathbf{x}_0|^2}$.

7. Use problem 6 to recover the two dimensional version of Poisson's formula for the ball that we derived in class using separation of variables.