Math 1272: Calculus II
Midterm I Review

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Evaluate

$$\int x \sec^2 x \, dx.$$

Integrate by parts: $u = x$, $du = dx$

$dv = \sec^2 x \, dx$, $v = \tan x \, dx$

$$\int x \sec^2 x \, dx = x \tan x - \int \tan x \, dx$$

$$= x \tan x - \int \frac{\sin x}{\cos x} \, dx$$

sub $u = \cos x$

$du = -\sin x \, dx$

$$= x \tan x + \int \frac{1}{u} \, du$$

$$= x \tan x + \ln|u| + C$$

$$= x \tan x + \ln|\cos x| + C$$

Evaluate

$$\int x \tan^2 x \, dx.$$

Use $1 + \tan^2 x = \sec^2 x$

$$\int x \tan^2 x \, dx = \int x(\sec^2 x - 1)\, dx$$

$$= \underbrace{\int x \sec^2 x}_{\text{same as problem 1}} - \int x \, dx$$

$$= x \tan x + \ln|\cos x| - \frac{1}{2}x^2 + C$$

Evaluate

$$\int \frac{\sqrt{x^2-1}}{x}\,dx.$$

Trig substitution $x = \sec\theta$, $dx = \tan\theta\sec\theta\,d\theta$

$$\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = \tan\theta$$

$$\int \frac{\sqrt{x^2-1}}{x}\,dx = \int \frac{\tan\theta \cdot \tan\theta \sec\theta}{\sec\theta}\,d\theta$$

$$= \int \tan^2\theta\,d\theta \qquad \tan^2 = \sec^2 - 1$$

$$= \int \sec^2\theta - 1 \, d\theta$$

$$= \tan\theta - \theta + C$$

Evaluate

$$\int \tan^3 \theta \sec^3 \theta \, d\theta.$$

Use substitution $u = \sec\theta$

$du = \tan\theta \sec\theta \, d\theta$

$$\int \tan^3 \sec^3 d\theta = \int \tan^2\theta \, u^2 \, du$$

$$= \int (\sec^2\theta - 1) u^2 \, du$$

$$= \int (u^2 - 1) u^2 \, du$$

$$= \int u^4 - u^2 \, du$$

$$= \frac{1}{5} u^5 - \frac{1}{3} u^3 + C$$

Determine if the following integrals are convergent or divergent. If convergent, evaluate the integral.

1.
$$\int_{\pi}^{\infty} \frac{x^2+1}{x-2}\,dx.$$

2.
$$\int_{1}^{2} \frac{2u}{\sqrt{u^2-1}}\,du.$$

3.
$$\int_{0}^{\infty} x^2 e^{-x}\,dx.$$

1. $\dfrac{x^2+1}{x-2} \geq \dfrac{x^2}{x-2} \geq \dfrac{x^2}{x} = x \quad , \; x \geq 2$

Since $\displaystyle\int_{\pi}^{t} x \, dx = \frac{1}{2}x^2\Big]_{\pi}^{t} = \frac{1}{2}(t^2-\pi^2)$

and $\displaystyle\lim_{t\to\infty} \frac{1}{2}(t^2-\pi^2) = \infty$

$\displaystyle\int_{\pi}^{\infty} x \, dx$ is <u>divergent</u>

Since $\dfrac{x^2+1}{x-2} \geq x \geq 0$ for $x \geq \pi$

$\displaystyle\int_{\pi}^{\infty} \frac{x^2+1}{x-2}\,dx$ is divergent

by Comparison Thm.

2. $\displaystyle\int_{1}^{2} \frac{2u}{\sqrt{u^2-1}}\,du.$

Singularity (blow up) as $u \longrightarrow 1^+$

So write $\displaystyle\int_t^2 \frac{2u}{\sqrt{u^2-1}}\,du$, $t>1$

Sub $v=u^2$, $dv=2u\,du$,

$$\int_t^2 \frac{2u}{\sqrt{u^2-1}}\,du = \int_{t^2}^4 \frac{1}{\sqrt{v-1}}\,dv$$

$$= 2\sqrt{v-1}\Big]_{t^2}^4$$

$$= 2\left(\sqrt{4-1} - \sqrt{t^2-1}\right)$$

$$= 2\left(\sqrt{3} - \sqrt{t^2-1}\right)$$

Since

$$\lim_{t \to 1^+} \int_t^2 \frac{2u}{\sqrt{u^2-1}}\, du = \lim_{t \to 1^+} 2\left(\sqrt{3} - \sqrt{t^2-1}\right)$$

$$= 2\sqrt{3}$$

The integral converges and

$$\int_1^2 \frac{2u}{\sqrt{u^2-1}}\,du = 2\sqrt{3}$$

3. $\displaystyle\int_0^\infty x^2 e^{-x}\,dx$. This integral converges. To compute the value

$$\int_0^t x^2 e^{-x}\,dx \overset{IBP}{=} -x^2 e^{-x}\Big]_0^t + \int_0^t 2x e^{-x}\,dx$$

$u = x^2$ $\quad du = 2x\,dx$

$dv = e^{-x}\,dx$ $\quad v = -e^{-x}$

$$= -t^2 e^{-t} + \int_0^t 2x e^{-x}\,dx$$

$$\overset{\text{IBP}}{=} -t^2 e^{-t} - \left[2x e^{-x}\right]_0^t + \int_0^t 2e^{-x}\,dx$$

$u = 2x \quad du = 2\,dx$

$dv = e^{-x}dx\,, \quad v = -e^{-x}$

$$= -t^2 e^{-t} - 2te^{-t} - \left[2e^{-x}\right]_0^t$$

$$= \underbrace{-t^2 e^{-t} - 2te^{-t} - 2e^{-t}}_{\to 0 \text{ as } t \to \infty} + 2$$

Hence $\displaystyle\int_0^\infty x^2 e^{-x}\,dx = 2.$

Evaluate the integral

$$\int \frac{3x^2 + 6x + 2}{x^3 + 3x^2 + 2x}\, dx.$$

Roots $x = \dfrac{-3 \pm \sqrt{9-8}}{2}$

$= -2, -1$

Partial fractions

$$x^3 + 3x^2 + 2x = x\overbrace{(x^2 + 3x + 2)}$$

$$= x(x+2)(x+1)$$

$$\frac{3x^2 + 6x + 2}{x^3 + 3x^2 + 2x} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x+1}$$

$$= \frac{A(x+2)(x+1) + Bx(x+1) + Cx(x+2)}{x^3+3x^2+2x}$$

$$= \frac{A(x^2+3x+2) + B(x^2+x) + C(x^2+2x)}{x^3+3x^2+2x}$$

$$= \frac{(A+B+C)x^2 + (3A+B+2C)x+2A}{x^3+3x^2+2x}$$

Equate like terms $(x^2, x, 1)$ on both sides to get

$$\left.\begin{aligned} A + B + C &= 3 \\ 3A + B + 2C &= 6 \\ 2A &= 2 \end{aligned}\right\} \Rightarrow$$

$A = 1$, $\quad B + C = 2$

$\quad\quad - (B + 2C = 3)$

$\quad\quad -C = -1$

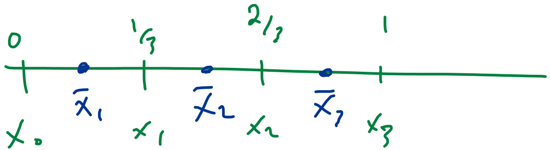
$B = 1$, $\quad C = 1$

$$\int \frac{3x^2 + 6x + 2}{x^3 + 3x^2 + 2x}\,dx = \frac{1}{x} + \frac{1}{x+2} + \frac{1}{x+1}$$

$$= \ln|x| + \ln|x+2| + \ln|x+1| + C$$

Use the midpoint rule with $n = 3$ to approximate $\int_0^1 x^3\,dx$.

$$\Delta x = \frac{1-0}{3} = \frac{1}{3}, \quad x_0 = 0,\ x_1 = \frac{1}{3},\ x_2 = \frac{2}{3},\ x_3 = 1$$



$$\bar{x}_1 = \frac{0+\frac{1}{3}}{2} = \frac{1}{6}$$

$$\bar{x}_2 = \frac{\frac{1}{3}+\frac{2}{3}}{2} = \frac{1}{2}$$

$$\bar{x}_3 = \frac{\frac{2}{3}+1}{2} = \frac{5}{6}$$

$$M_3 = \Delta x\left(\left(\frac{1}{6}\right)^3 + \left(\frac{1}{2}\right)^3 + \left(\frac{5}{6}\right)^3\right)$$

$$= \frac{1}{3}\left(\frac{1+3^3+5^3}{6^3}\right)$$

I would make this easier on exam.

$$= \frac{1}{3}\left(\frac{1+27+125}{216}\right)$$

$$= \frac{1}{3}\left(\frac{153}{216}\right) = \frac{153}{648}$$

Note exact value $\int_0^1 x^3\,dx = \frac{1}{4} = \frac{150}{600}$

Find the center of mass of the region $\mathcal{R}$ in the plane bounded by the curves

$$x = 0, \; x = 1, \; y = 0, \quad \text{and } y = \sqrt{x^2+1}.$$

$$\text{Area} = A = \int_0^1 \sqrt{x^2+1}\, dx \qquad x = \tan\theta \quad dx = \sec^2\theta\, d\theta$$

$$= \int_0^{\pi/4} \sec^3\theta\, d\theta$$

$\sec\left(\frac{\pi}{4}\right) = \sqrt{2}$

$\tan\left(\frac{\pi}{4}\right) = 1$

$\tan(0) = 0$

$$= \left[\frac{1}{2}\tan\theta\sec\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|\right]_0^{\pi/4}$$

$$= \left[\frac{1}{2}\sqrt{2} + \frac{1}{2}\ln|\sqrt{2}+1|\right]$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{2} \ln\left(\sqrt{2}+1\right)$$

$$\bar{x} = \frac{1}{A} \int_0^1 x\sqrt{x^2+1}\; dx$$

sub $u = x^2$

$du = 2x\, dx$

$x\, dx = \frac{1}{2} du$

$$= \frac{1}{A} \int_0^1 \sqrt{u+1}\; \frac{1}{2} du$$

$$= \frac{1}{2A} \cdot \frac{2}{3} (u+1)^{3/2} \Big]_0^1 = \frac{1}{3A}\left(2^{3/2} - 1\right)$$

$$= \frac{1}{2A}\left(2\sqrt{2} - 1\right)$$

$$\bar{y} = \frac{1}{A}\int_0^1 \frac{1}{2}(x^2+1)\,dx$$

$$= \frac{\sqrt{2}}{A} - \frac{1}{2A}$$

$$= \frac{1}{2A}\left(\frac{1}{3}x^3 + x\right)\Big]_0^1 = \frac{1}{2A}\left(\frac{1}{3}+1\right)$$

$$= \frac{1}{2A}\left(\frac{4}{3}\right) = \frac{2}{3A}.$$

Find the arclength of the curve

$$y = \ln(\sec x), \quad 0 \le x \le \cancel{\pi/2}. \quad \pi/4$$

$$\frac{dy}{dx} = \frac{1}{\sec x} \cdot \frac{d}{dx} \sec x = \frac{\sec x \tan x}{\sec x} = \tan x$$

$$L = \int_0^{\frac{\pi}{4}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^{\frac{\pi}{4}} \sqrt{1 + \tan^2 x}\, dx$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{\sec^2 x}\, dx$$

$$= \int_0^{\frac{\pi}{4}} \sec x \, dx$$

$$= \ln|\sec x + \tan x|\Big]_0^{\frac{\pi}{4}}$$

$$= \ln|\sqrt{2} + 1| - \ln(1)$$

$$= \ln(\sqrt{2} + 1).$$

Find the area of the surface of revolution obtained by rotating the curve

$$y = \sin x, \quad 0 \leq x \leq \pi/2$$

about the x-axis. You may recall

$$\int \sec^3 \theta \, d\theta = \frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| + C.$$

$$A = \int_0^{\frac{\pi}{2}} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$$= \int_0^{\frac{\pi}{2}} 2\pi \sin x \sqrt{1 + \cos^2 x} \, dx$$

Sub $u = \cos x$, $du = -\sin x \, dx$

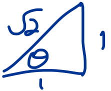
$$= \int_1^0 -2\pi \sqrt{1+u^2}\, du$$

$$= \int_0^1 2\pi \sqrt{1+u^2}\, du$$

sub $u = \tan\theta$

$du = \sec^2\theta \, d\theta$

$$= \int_0^{\pi/4} 2\pi \sqrt{1+\tan^2\theta}\, \sec^2\theta \, d\theta$$



$\theta = \frac{\pi}{4}$

$$= \int_0^{\frac{\pi}{4}} 2\pi \sec^3\theta \, d\theta$$

$\tan\frac{\pi}{4} = 1$

$\sec\frac{\pi}{4} = \frac{1}{\cos\frac{\pi}{4}} = \sqrt{2}$

$$= 2\pi \cdot \frac{1}{2}\left[\tan\theta\sec\theta + \ln|\tan\theta + \sec\theta|\right]\Big|_0^{\pi/4}$$

$$= \pi\left[\sqrt{2} + \ln(1+\sqrt{2}) - (0 + \ln(1))\right]$$

$$= \pi\left(\sqrt{2} + \ln(1+\sqrt{2})\right)$$