Math 1272: Calculus II
9.4 Models for population growth

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Law of natural growth

$$\frac{dP}{dt} = kP$$

Law of natural growth

$$\frac{dP}{dt} = kP \quad P(0) = P_0.$$

Solution is

$$P(t) = P_0 e^{kt}.$$

The Logistic Model

$$\boxed{\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right).}$$

Notice:

1. $\frac{dP}{dt} \approx kP$ if P is small.

2. $\frac{dP}{dt} < 0$ if $P > M$.

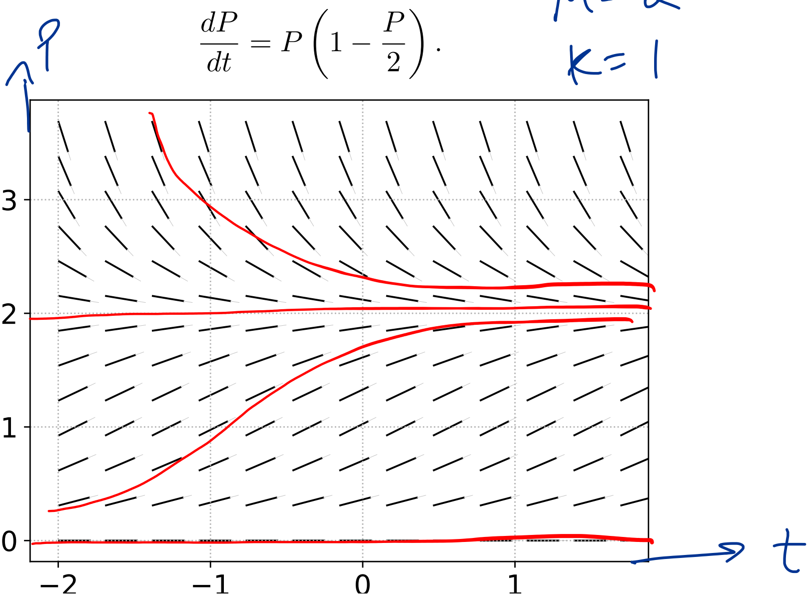
Equillibrium

$P = 0$

$P = 2$

$M = 2$

$k = 1$



Solving the Logistic equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right), \quad P(0) = P_0.$$

Separate variables

$$\int \frac{dP}{kP\left(1-\frac{P}{M}\right)} = \int dt = t + C$$

↑ use partial fractions

Same as

$$\frac{1}{k}\int \frac{M}{P(M-P)}\,dP$$

$$\frac{M}{P(M-P)} = \frac{A}{P} + \frac{B}{M-P}$$

$$M = P(M-P)\left(\frac{A}{P} + \frac{B}{M-P}\right)$$

$$= (M-P)A + PB$$

$$= (B-A)P + AM$$

$B - A = 0$ $\qquad$ $\underbrace{A = 1}$, $B = 1$

$$\frac{1}{k}\int \frac{M}{P(M-P)}\,dP = \frac{1}{k}\int \frac{1}{P} + \frac{1}{M-P}\,dP$$

$$= \frac{1}{k}\left(\ln P - \ln|M-P|\right) + C$$

$$\underbrace{= \frac{1}{k}\ln\left(\frac{P}{|M-P|}\right) + C}_{= t + C}$$

$$\frac{1}{k}\ln\left(\frac{P}{|M-P|}\right) = t + C_1$$

$$\ln\left(\frac{P}{|M-P|}\right) = kt + C_2$$

$A_1 = e^{C_2}$

$$\frac{P}{|M-P|} = e^{kt+C_2} = A_1 e^{kt}$$

$A = \pm A_1$

$$\frac{P(t)}{M-P(t)} = A e^{kt}$$

$$P = (M-P)Ae^{kt} = MAe^{kt} - PAe^{kt}$$

$$P + PAe^{kt} = MAe^{kt}$$

$$P(1+Ae^{kt}) = MAe^{kt}$$

$$P(t) = \frac{MA\cancel{e^{kt}}}{1+A\cancel{e}^{kt}} \cdot \frac{e^{-kt}}{e^{-kt}}$$

$$\boxed{P(t) = \frac{MA}{A+e^{-kt}}}$$

A arbitrary
depends on P_0

Write down the solution of the initial value problem

$$\frac{dP}{dt} = 0.05P\left(1 - \frac{P}{500}\right), \quad P(0) = 100,$$

and use it to estimate the population sizes $P(40)$ and $P(80)$. At what time does the population reach 400?

$$P(t) = \frac{MA}{A + e^{-kt}}$$

$k = 0.05$

$M = 500$

$P_0 = 100$

$$P(t) = \frac{500A}{A + e^{-0.05t}}$$

$$100 = P(0) = \frac{500A}{A+1} \rightsquigarrow \text{solve for } A$$

$$\cancel{100}(A+1) = 5\cancel{00}A$$

$$A+1 = 5A, \quad 4A = 1, \quad A = \frac{1}{4}$$

$$P(t) = \frac{500\left(\frac{1}{4}\right)}{\frac{1}{4} + e^{-0.05t}} \cdot \frac{4}{4}$$

$$= \frac{500}{1+4e^{-0.05t}}$$

$P(40)$, $P(80)$ plus in $t=40$, $t=80$

When does population reach 400?

$$400 = \frac{500}{1+4e^{-0.05t}}$$

$$400\left(1+4e^{-0.05t}\right) = 500$$

$$1+4e^{-0.05t} = \frac{5}{4}$$

$$4e^{-0.05t} = \frac{1}{4}$$

$$e^{-0.05t} = \frac{1}{16}$$

$$-0.05t = \ln\left(\frac{1}{16}\right)$$

$$t = -\frac{1}{0.05} \ln\left(\frac{1}{16}\right)$$

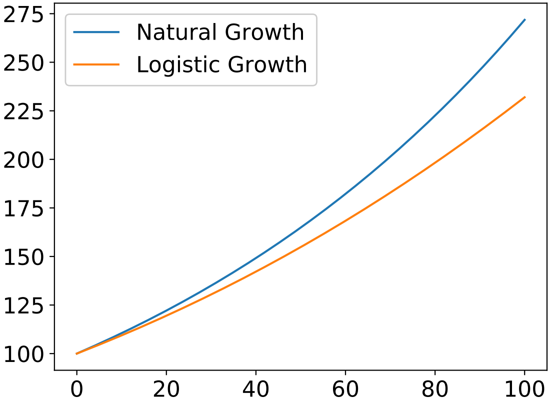
$$0.05 = \frac{5}{100} = \frac{1}{20}$$

$$= \frac{\ln(16)}{0.05}$$

$$\boxed{t = 20 \ln(16)}$$

Comparison of Natural and Logistic Growth

$$\frac{dP}{dt} = 0.01P \quad \text{and} \quad \frac{dP}{dt} = 0.01P\left(1 - \frac{P}{1000}\right), \quad P(0) = 100.$$



$k = 0.01$

$M = 1000$

P small

$\frac{dP}{dt} \approx kP$

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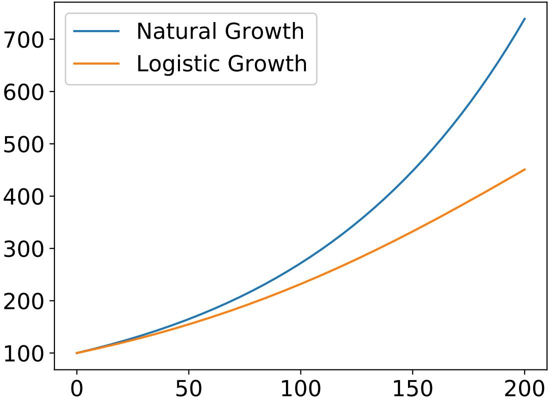

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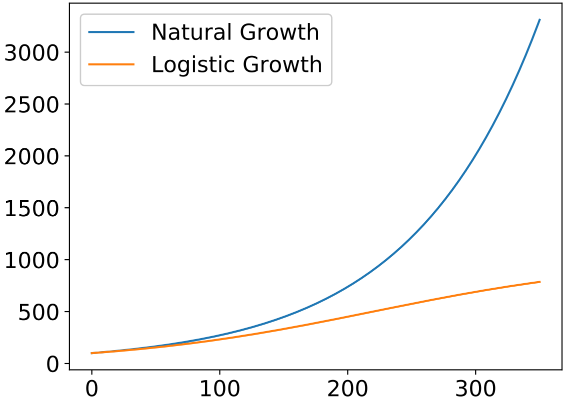

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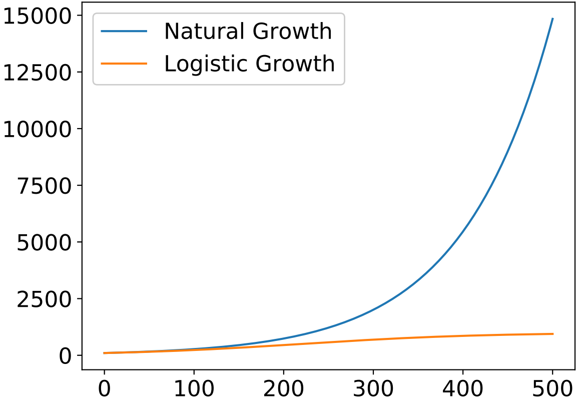

Other models for population growth

1. Logistic with harvesting

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) - c.$$

2. Minimum population $0 < m < M$

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)\underbrace{\left(1 - \frac{m}{P}\right)}_{<0 \text{ if } P < m}.$$

$\frac{m}{P} > 1$

$\lim_{t \to \infty} P(t) = 0$

if $P(0) < m$