Math 1272: Calculus II
9.5 Linear Equations

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Linear equations

A **fist order linear** differential equation has the form

first

$$\frac{dy}{dx} + P(x)y = Q(x).$$

We can solve all such equations with the *integrating factor* method.

Integrating factor

$$y' + P(x)y = Q(x)$$

Idea: Multiply both sides by $I(x)$

$$\frac{d}{dx}\big(I(x)y\big) = I(x)\big(y' + P(x)y\big) = I(x)Q(x)$$

Choose I so this holds

$$I(x)y(x) = \int I(x)Q(x)\,dx$$

$$y(x) = \frac{1}{I(x)}\int I(x)Q(x)\,dx$$

How to select integrating factor $I(x)$?

$$\frac{d}{dx}\left(I(x)y\right) = I(x)\left(y' + P(x)y\right)$$

$\|$ want

$$\frac{d}{dx}\left(I(x)y\right) = I(x)y' + I'(x)y$$

Need

$$I(x)P(x) = I'(x) = \frac{dI}{dx}$$

$$\int P(x)\,dx = \int \frac{dI}{I}$$

$$\ln|I| = \int p(x)\,dx$$

$$|I| = e^{\int p(x)\,dx}$$

$$I(x) = \pm\, e^{\int p(x)\,dx} \qquad \text{Chmsc} +$$

$$I(x) = e^{\int p(x)\,dx}$$

$$y(x) = \frac{1}{I(x)} \int I(x)\,Q(x)\,dx$$

Solve the differential equation

$$\frac{dy}{dx} + 3x^2y = 6x^2.$$

$P(x) = 3x^2$

$Q(x) = 6x^2$

Integrating factor $I(x) = e^{\int P(x)\,dx} = e^{x^3}$

$$\underbrace{e^{x^3}y' + 3x^2e^{x^3}y}_{\frac{d}{dx}(e^{x^3}y)} = 6x^2e^{x^3}$$

$$\frac{d}{dx}\left(e^{x^3}y\right) = 6x^2e^{x^3}$$

$$e^{x^3} y = \int 6x^2 e^{x^3}\,dx$$

$$u = x^3, \quad du = 3x^2\,dx$$

$$= 2\int e^u\,du$$

$$= 2e^u + C$$

$$= 2e^{x^3} + C$$

$$\boxed{y(x) = 2 + Ce^{-x^3}}$$

Find the solution of the initial value problem

$$x^2y' + xy = 1, \quad x > 0, \quad y(1) = 2.$$

$P(x) = \frac{1}{x}$

$Q(x) = \frac{1}{x^2}$

Standard form $\quad y' + \frac{1}{x}y = \frac{1}{x^2}$

Integrating factor $\quad I(x) = e^{\int P(x)\,dx} = e^{\ln x} = x$

$$x\left(y' + \frac{1}{x}y\right) = \frac{x}{x^2}$$

$$\boxed{\begin{array}{l}\text{LHS} \\ \frac{d}{dx}(Iy)\end{array}}$$

$$xy' + y = \frac{1}{x}$$

$$\frac{d}{dx}(xy) = \frac{1}{x}$$

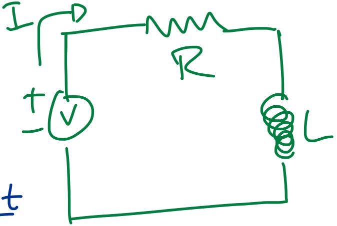
$$xy = \ln x + C, \quad x > 0$$

$$\begin{cases} y(x) = \dfrac{\ln x}{x} + \dfrac{C}{x} \\ y(1) = 2 \end{cases}$$

$$2 = y(1) = \frac{C}{1} = C \rightsquigarrow C = 2$$

$$\boxed{y(x) = \frac{\ln x}{x} + \frac{2}{x}}$$

The current $I(t)$ in an electric circut with voltage source V, resistance R and inductance L satisfies the differential equation



$$L\frac{dI}{dT} + RI(t) = V.$$

dt

Solve for the current $I(t)$.

$P(t)$ $\quad \int P(t)\,dt = \int \frac{R}{L}\,dt = \frac{Rt}{L}$

$$I' + \frac{R}{L} I = \frac{V}{L}$$

Integrating factor

$$J(t) = e^{\int P(t)\,dt} = e^{\frac{R}{L}t}$$

$$e^{\frac{R}{L}t}\left(I' + \frac{R}{L} I\right) = \frac{V}{L} e^{\frac{R}{L}t}$$

Check

$$\frac{d}{dt}\left(e^{\frac{R}{L}t} I\right) = \frac{V}{L} e^{\frac{R}{L}t}$$

$$e^{\frac{R}{L}t} I(t) = \int \frac{V}{L} e^{\frac{R}{L}t}\, dt$$

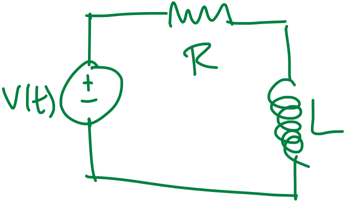
$$= \frac{V}{\cancel{L}} \cdot \frac{\cancel{L}}{R} e^{\frac{R}{L}t} + C$$

$$= \frac{V}{R} e^{\frac{R}{L}t} + C$$

$$\boxed{I(t) = \frac{V}{R} + C e^{-\frac{R}{L}t}}$$

Suppose the voltage V is variable (AC or alternating current), given by $V(t) = \sin(t)$. The current $I(t)$ then satisfies

$$L\frac{dI}{dT} + RI(t) = \sin(t).$$



Solve for the current $I(t)$.

$$I' + \frac{R}{L} I = \frac{\sin(t)}{L} \quad , \quad J(t) = e^{\frac{R}{L}t}$$

$$e^{\frac{R}{L}t}\left(I' + \frac{R}{L} I\right) = \frac{\sin(t)}{L} e^{\frac{R}{L}t}$$

$$\frac{d}{dt}\left(e^{\frac{R}{L}t} I\right) = \frac{\sin(t)}{L} e^{\frac{R}{L}t}$$

$$e^{\frac{R}{L}t} I(t) = \int \frac{\sin(t)}{L} e^{\frac{R}{L}t}\, dt$$

$$\int \frac{\sin(t)}{L} e^{\frac{R}{L}t}\, dt = -\frac{\cos t}{L} e^{\frac{R}{L}t} + \int \cos t \frac{R}{L^2} e^{\frac{R}{L}t} dt$$

$u = \frac{e^{\frac{R}{L}t}}{L}$, $dv = \sin(t)dt$

$du = \frac{R}{L^2} e^{\frac{R}{L}t} dt$,

$v = -\cos t$

$u = \frac{R}{L^2} e^{\frac{R}{L}t}$, $dv = \cos t\, dt$

$du = \frac{R^2}{L^3} e^{\frac{R}{L}t}$

$v = \sin t$

$$= -\frac{\cos t\, e^{\frac{R}{L}t}}{L} + \frac{R}{L^2} \sin t\, e^{\frac{R}{L}t} - \int \sin t \frac{R^2}{L^3} e^{\frac{R}{L}t} dt$$

Add to LHS $\rightarrow$

$$\left(\frac{R^2}{L^2}+1\right)\frac{1}{L}\int \sin t\, e^{\frac{R}{L}t}\,dt = -\frac{\cos t\, e^{\frac{R}{L}t}}{L} + \frac{R}{L^2}\sin t\, e^{\frac{R}{L}t}$$

$$\frac{1}{L}\int \sin t\, e^{\frac{R}{L}t}\,dt = \frac{1}{\frac{R^2}{L^2}+1}\left[-\frac{\cos t\, e^{\frac{R}{L}t}}{L} + \frac{R}{L^2}\sin t\, e^{\frac{R}{L}t}\right]+C$$

$$e^{\frac{R}{L}t}I(t) = \int \frac{\sin(t)}{L}e^{\frac{R}{L}t}\,dt$$

$$\boxed{I(t) = \frac{1}{\frac{R^2}{L^2}+1}\left[-\frac{\cos t}{L} + \frac{R}{L^2}\sin t\right] + Ce^{-\frac{R}{L}t}}$$

$$I(t) = \frac{1}{R^2+L^2}\left(-L\cos t + R\sin t\right) + Ce^{-\frac{R}{L}t}$$