Math 1272: Calculus II
10.1 Curves defined by parametric equations

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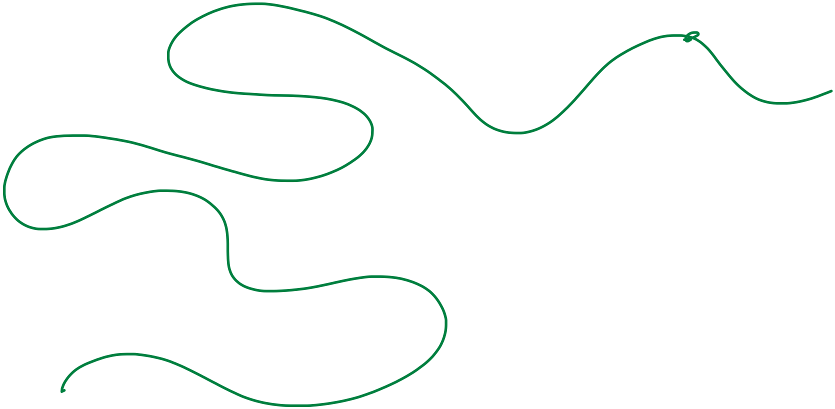
Parametric equations

We have seen curves of the form $(x, f(x))$. Some curves cannot be expressed this way.

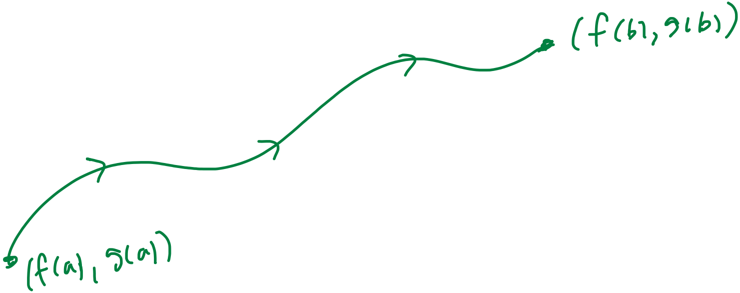


Parametric equations

A curve in **parametric form** is given by

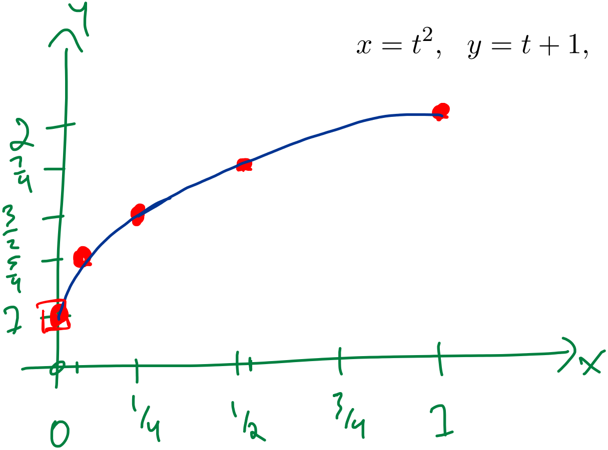
$$x = f(t), \quad y = g(t) \quad \text{for } a \le t \le b.$$

The **initial point** is $(f(a), g(a))$ and the **terminal point** is $(f(b), g(b))$.



Sketch and identify the curve defined by

$$x = t^2, \quad y = t + 1, \quad 0 \le t \le 1.$$



t	x	y
0	0	1
1/4	1/16	5/4
1/2	1/4	3/2
3/4	9/16	7/4
1	1	2

Eliminate t: $t = y - 1$, $x = t^2 = (y-1)^2$

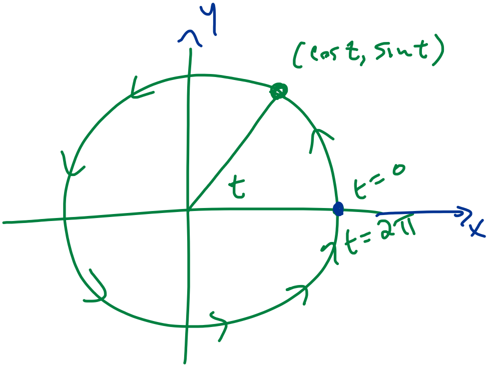
$$x = (y-1)^2$$

What curve is represented by the parametric equation

$$x = \cos t, \quad y = \sin t, \quad 0 \le t \le 2\pi?$$

$$x^2 + y^2 = (\cos t)^2 + (\sin t)^2 = 1$$

Circle of radius one.



$$\frac{dy}{dt} = \cos(t)$$

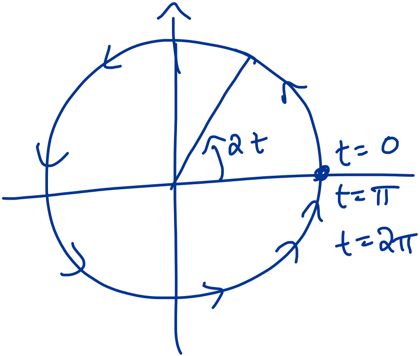
$$\left.\frac{dy}{dt}\right|_{t=0} = \cos(0) = 1$$

What about

$$x = \cos(2t), \quad y = \sin(2t), \quad 0 \le t \le 2\pi?$$

$$x^2 + y^2 = \cos^2(2t) + \sin^2(2t) = 1$$

Completes the circle twice.



Find the parametric equation for a circle of radius r and center (h, k).

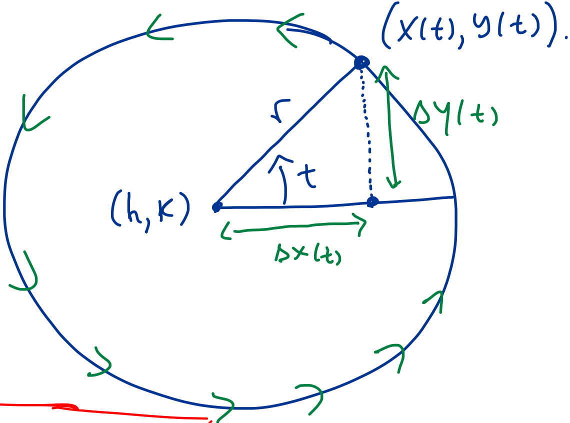
Equation for circle $\quad (x-h)^2 + (y-k)^2 = r^2$

$x(t) = h + \Delta x(t)$

$y(t) = k + \Delta y(t)$

$\cos(t) = \dfrac{\Delta x(t)}{r}$

$\sin(t) = \dfrac{\Delta y(t)}{r}$



$$x(t) = h + r\cos t, \quad 0 \le t \le 2\pi$$
$$y(t) = k + r\sin t$$

Check

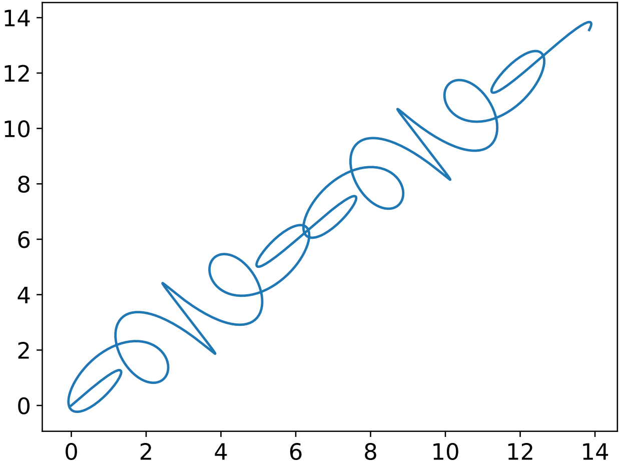
$$(x(t)-h)^2 + (y(t)-k)^2$$

$$= (r\cos t)^2 + (r\sin t)^2$$

$$= r^2\cos^2 t + r^2\sin^2 t$$

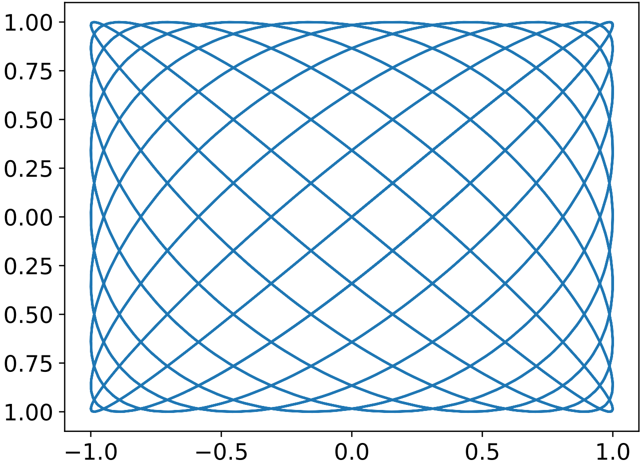
$$= r^2(\cos^2 t + \sin^2 t) = r^2$$

Examples of parametric curves



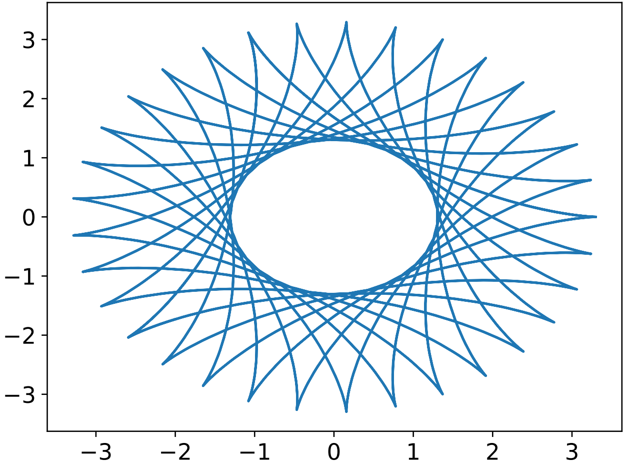
$$x = t + \sin(5t), \quad y = t + \sin(6t).$$

Examples of parametric curves



$$x = \sin(9t), \quad y = \sin(10t).$$

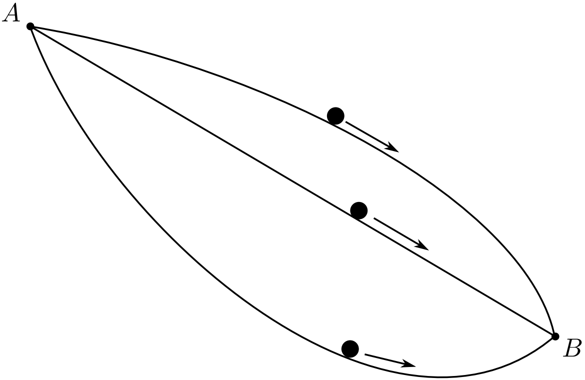
Examples of parametric curves



$$x = 2.3\cos(10t) + \cos(23t), \quad y = 2.3\sin(10t) - \sin(23t).$$

Brachistochrone curve

Question: What is the shortest path for a rolling ball?



Brachistochrone demo:

- https://www.youtube.com/watch?v=0KjUqPps8vM

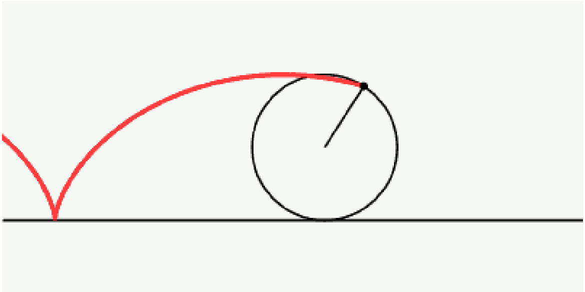
Brachistochrone curve

The Brachistochrone curve $(x, y(x))$ satisfies the differential equation

$$y(x) + y(x)y'(x)^2 = C.$$

$y\left(1+\left[\frac{dy}{dx}\right]^2\right) = C$

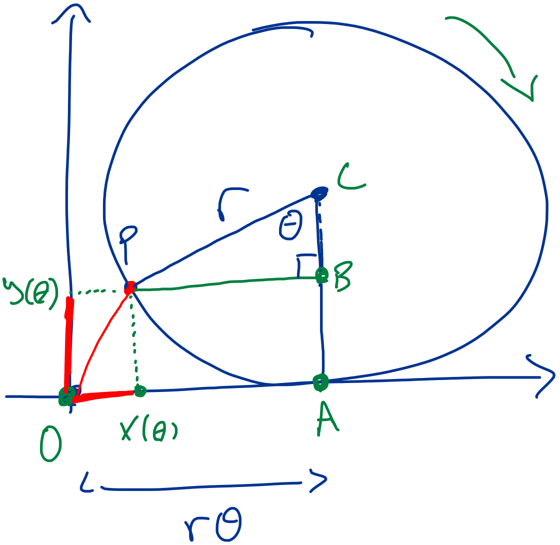
The solution is a cycloid (https://en.wikipedia.org/wiki/Cycloid):



$\sin\theta = \frac{|PB|}{r}, \quad \cos\theta = \frac{|BC|}{r}$

Cycloids

Find the parametric equation for a cycloid.



$P = (x(\theta), y(\theta))$

$|OA| = \text{arc}(PA) = r\theta$

$x(\theta) = |OA| - |PB|$

$= r\theta - r\sin\theta$

$y(\theta) = |AC| - |BC|$

$= r - r\cos\theta$

$$x(\theta) = r(\theta - \sin\theta), \quad y(\theta) = r(1 - \cos\theta)$$

Recall: Bracistichrone curve solves

$$y(x)\left(1 + y'(x)^2\right) = C$$

$$y' = \frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cancel{r}\sin\theta}{\cancel{r}(1-\cos\theta)}$$

$$1 + y'(x)^2 = 1 + \frac{\sin^2\theta}{(1-\cos\theta)^2} = \frac{(1-\cos\theta)^2 + \sin^2\theta}{(1-\cos\theta)^2}$$

$$x(\theta) = r(\theta - \sin\theta), \quad y(\theta) = r(1-\cos\theta)$$

$$\frac{dx}{d\theta} = r(1-\cos\theta) = y(\theta), \quad \frac{dy}{d\theta} = r\sin\theta$$

$$1 + y'(x)^2 = \frac{(1-\cos\theta)^2 + \sin^2\theta}{(1-\cos\theta)^2}$$

$$= \frac{1 - 2\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{1}}{(1-\cos\theta)^2}$$

$$= \frac{2(1-\cos\theta)}{(1-\cos\theta)^2} = \frac{2}{1-\cos\theta}$$

$$y\left(1+\left(\frac{dy}{dx}\right)^2\right) = \underbrace{r(1-\cos\theta)}_{y(\theta)}\left(\frac{2}{1-\cos\theta}\right) = 2r$$