Math 1272: Calculus II
7.1 Integration by parts

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Integration by parts

Integration analog of the **product rule**

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x).$$

Integrate both sides

$$f(x)g(x) = \int f(x)g'(x) + g(x)f'(x)\,dx$$

$$\boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx}$$

$$(*) \quad \int \underline{f(x)}\, g'(x)\, dx = f(x)\, g(x) - \int g(x)\, \underline{f'(x)}\, dx$$

Write $u = f(x)$, $v = g(x)$

$$\frac{du}{dx} = f'(x), \quad \frac{dv}{dx} = g'(x)$$

$$du = f'(x)\, dx, \quad dv = g'(x)\, dx$$

$$(*) \quad \boxed{\int u\, dv = uv - \int v\, du}$$

Integration by parts

(1)
$$\boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx.}$$

(2)
$$\boxed{\int u\,dv = uv - \int v\,du.}$$

Goal: Make integral on right hand side **simpler** than what we started with.

Example 1. Find $\int x \cos x \, dx$.

$$f(x) = x$$

$$g'(x) = \cos(x)$$

$$f'(x) = 1$$

$$g(x) = \sin(x)$$

$$\int f(x) g'(x) \, dx = f(x) g(x) - \int g(x) f'(x) \, dx$$

$$\int x \cos(x) \, dx = x \sin(x) - \int \sin(x) \, dx$$

$$= x \sin(x) + \cos(x) + C$$

Find $\int x\cos(x)\,dx$. $\left[\int u\,dv = uv - \int v\,du\right]$

$dv = \cos(x)\,dx$, $u = x \rightsquigarrow \boxed{\frac{du}{dx} = 1}$

$\frac{dv}{dx} = \cos(x) \implies v = \sin(x)$

$$\int x\cos(x)\,dx = \int u\,dv$$

$$= uv - \int v\,du$$

$$= x\sin(x) - \int \sin(x)\,dx$$

Example 2. Find $\int \ln s \, ds$. $= \int 1 \ln(s) \, ds$

$$u = \ln(s), \quad dv = 1 \, ds \rightsquigarrow \frac{dv}{ds} = 1$$

$$\frac{du}{ds} = \frac{1}{s} \implies du = \frac{ds}{s} \qquad \hookrightarrow v = s$$

$$\int \ln(s) \, ds = \int u \, dv = uv - \int v \, du$$

$$= \ln(s) s - \int 1 \, ds$$

$$= \ln(s) s - s + C$$

Example 3. Find $\int e^y \cos y \, dy$.

$$\int u\,dv = uv - \int v\,du.$$

$u = e^y$, $dv = \cos y \, dy \longrightarrow \dfrac{dv}{dy} = \cos y$

$\dfrac{du}{dy} = e^y \longrightarrow du = e^y dy$ | or $v = \sin y$

$$\int e^y \cos y \, dy = \int u\,dv = uv - \int v\,du$$

$$= e^y \sin y - \int \sin y \, e^y dy$$

$$\int e^y \cos y\, dy = e^y \sin y - \int \sin y\, e^y\, dy$$

$$u = e^y \longrightarrow du = e^y\, dy$$

$$dv = \sin y\, dy \longrightarrow \frac{dv}{dy} = \sin y, \quad v = -\cos y$$

$$\int \sin y\, e^y\, dy = \int u\, dv = uv - \int v\, du$$

$$= -e^y \cos y + \int e^y \cos y\, dy$$

Go to (*)

Example 4. The **gamma function** $\Gamma(n)$ is defined by

$$\Gamma(n) = \int_0^\infty \underbrace{e^{-x}x^{n-1}\,dx} = \lim_{T\to\infty}\int_0^T e^{-x}x^{n-1}\,dx.$$

Show that for positive integers n

$$\Gamma(n) = n! = n\cdot(n-1)\cdot(n-2)\cdots 3\cdot 2\cdot 1.$$

$(n-1)!$

$u = x^{n-1} \rightsquigarrow \frac{du}{dx} = (n-1)x^{n-2}$

$dv = e^{-x}\,dx$ $\longrightarrow$ $\boxed{du = (n-1)x^{n-2}\,dx}$

$\frac{dv}{dx} = e^{-x} \rightsquigarrow \boxed{v = -e^{-x}}$

$\int u\,dv = uv - \int v\,du$

$$\int_0^T e^{-x}x^{n-1}\,dx = \underbrace{\cancel{-x^{n-1}e^{-x}\Big|_0^T}}_{0} - \int_0^T -e^{-x}(n-1)x^{n-2}\,dx$$

Remember limit $T \to \infty$

$$\underbrace{\int_0^\infty e^{-x}x^{n-1}\,dx}_{\Gamma(n)} = (n-1)\underbrace{\int_0^\infty e^{-x}x^{n-2}\,dx}_{\Gamma(n-1)}$$

$$\Gamma(n) = (n-1)\,\Gamma(n-1)$$
$$= (n-1)(n-2)\,\Gamma(n-2)$$

$$\vdots$$

$$= (n-1)(n-2)\cdots(2)(1)\Gamma(1)$$

$1, 1$

<u>Claim</u> : $\Gamma(1) = \int_0^\infty e^{-x}\,dx = 1$

So $\Gamma(n) = (n-1)!$

Linearity of integral

$$\int A f(x)\,dx = A\int f(x)\,dx$$

$$\int f(x) + g(x)\, dx = \int f(x)\, dx + \int g(x)\, dx$$

Definite integrals $\int_a^b f(x)\,dx$

If $F'(x) = f(x)$ then

$$\int_a^b f(x)\,dx = F(x)\Big|_{x=a}^{x=b}$$

$$= F(b) - F(a)$$

Fundamental theorem of Calculus

Definite integrals

Combining with the **fundamental theorem of calculus** we have

$$\boxed{\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b g(x)f'(x)\,dx.} \tag{3}$$

Example 5. Compute $\int_0^\pi t^2 \cos t\,dt$. $f(t) = t^2$, $f'(t) = 2t$

$g'(t) = \cos t$, $g(t) = \sin t$

$$\int_0^\pi t^2 \cos t\,dt = \underbrace{t^2 \sin t\Big]_0^\pi}_{=0} - \int_0^\pi \sin t \, 2t\,dt$$

$$= -2\boxed{\int_0^\pi t \sin t\,dt} = \pi$$

$f(t) = t, \quad f'(t) = 1, \quad g'(t) = \sin t$

$g(t) = -\cos t$

$$\int_0^\pi t \sin t \, dt = \underbrace{-t\cos t}_{fg}\Big]_0^\pi - \underbrace{\int_0^\pi -\cos t \, dt}_{gf'}$$

$\boxed{\cos \pi = -1}$

$$= \pi + \int_0^\pi \cos t \, dt$$

$$= \pi + \cancel{\sin t\Big]_0^\pi}^{\,0} = \pi$$

(*)

$$\int e^y \cos y \, dy = e^y \sin y - \int \sin y \, e^y \, dy$$

$u = e^y \longrightarrow du = e^y \, dy$

$dv = \sin y \, dy \longrightarrow \frac{dv}{dy} = \sin y, \quad v = -\cos y$

$$\int \sin y \, e^y \, dy = \int u \, dv = uv - \int v \, du$$

$$= -e^y \cos y + \int e^y \cos y \, dy$$

$$\int e^y \cos y \, dy = e^y \sin y - \int \sin y \, e^y \, dy$$

$$= e^y \sin y - \left[-e^y \cos y + \int e^y \cos y \, dy \right]$$

$$= e^y \sin y + e^y \cos y - \int e^y \cos y \, dy$$

$$\boxed{\int e^y \cos y \, dy = \frac{e^y}{2}(\sin y + \cos y)}$$

$$+ C$$