Math 1272: Calculus II
10.2 Calculus with parametric curves

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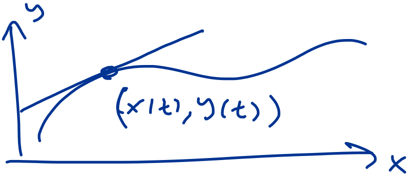
Tangents to parametric curves



Slope of tangent line to $(x(t), y(t))$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{\frac{dx}{dt}}$$

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}}$$

Second derivatives

$$\frac{d}{dx}(y) = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(y)}{\frac{dx}{dt}}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}} = \frac{\overbrace{\frac{d}{dt}\left[\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right]}^{\text{quotient rule}}}{\frac{dx}{dt}}$$

A curve C is defined by $x = t^2, y = t^3 - 3t$.

- Find the tangents at $(3, 0)$.
- Find the points where the tangent is horizontal or vertical. $\frac{dy}{dx}$
- Determine where the curve is concave up or concave down. $\frac{d^2y}{dx^2}$

$$\frac{dx}{dt} = 2t \quad , \quad \frac{dy}{dt} = 3t^2 - 3$$

$$3 = t^2 \quad , \quad 0 = t^3 - 3t$$

$$t = \pm\sqrt{3}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 3}{2t}$$

$$\underline{t = \sqrt{3}} \quad , \quad \frac{dy}{dx} = \frac{3 \cdot 3 - 3}{2\sqrt{3}} = \frac{6}{2\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

$t = -\sqrt{3}$, $\frac{dy}{dx} = \frac{3 \cdot 3 - 3}{2(-\sqrt{3})} = -\sqrt{3}$

Horizontal/Vertical

$$\frac{dy}{dx} = \frac{3t^2 - 3}{2t}$$

Vertical: $t = 0$

$x = t^2$

$y = t^3 - 3t$

$\boxed{(0,0)}$

Horizontal: $3t^2 - 3 = 0$

$t^2 = 1$, $t = \pm 1$

$x(1) = (1)^2 = 1$

$y(1) = 1 - 3 = -2$

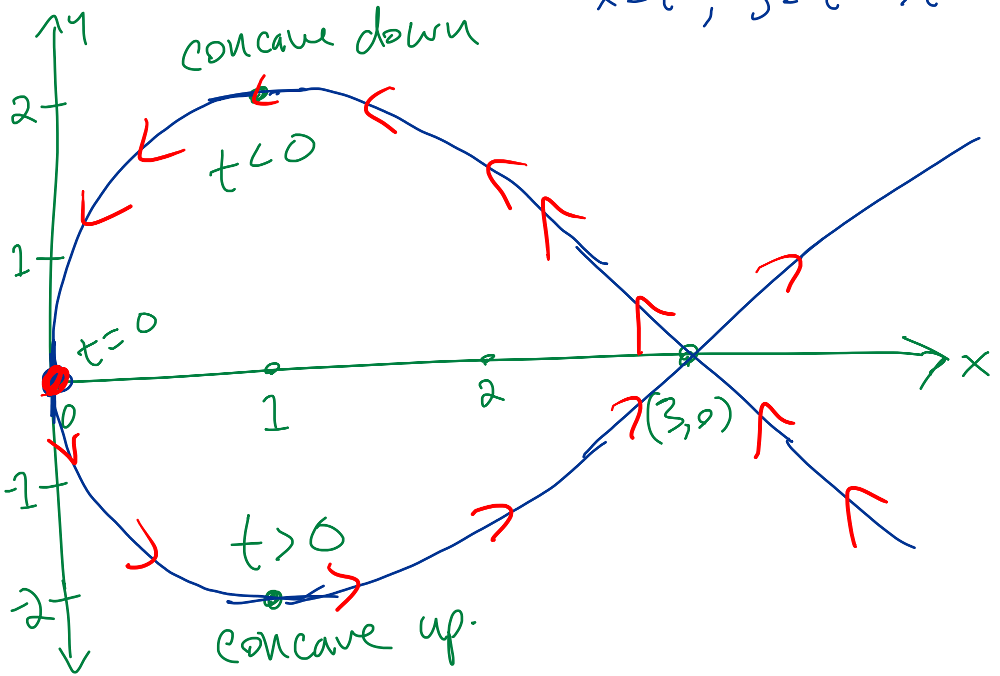
$\boxed{(1,-2)}$

$x(-1) = (-1)^2 = 1$

$y(-1) = -1 - (-3) = 2$

$\boxed{(1,2)}$

$x = t^2, \; y = t^3 - 3t$



Concave up/down

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$$\frac{dy}{dx} = \frac{3t^2-3}{2t}$$

$$\frac{dx}{dt} = 2t$$

$$\frac{dy}{dt} = 3t^2-3$$

$$= \frac{\frac{d}{dt}\left(\frac{3t^2-3}{2t}\right)}{2t}$$

$$= \frac{\frac{2t(6t)-(3t^2-3)2}{(2t)^2}}{2t}$$

$$(2t)^3 = 2^3 t^3$$
$$= 8t^3$$

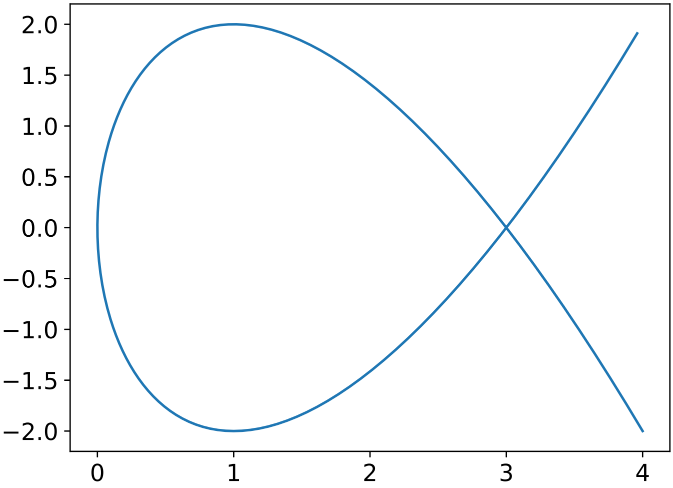
$$= \frac{12t^2 - 6t^2 + 6}{(2t)^3}$$

$$= \frac{6t^2 + 6}{8t^3}$$

$$= \frac{3t^2 + 3}{4t^3} = \underbrace{\frac{3}{4}\left(\frac{t^2+1}{t^3}\right)}_{\frac{d^2y}{dx^2}}$$

Concave up $t > 0$

Concave down $t < 0$



2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
0
1
2
3
4

Find the tangent to the cycloid $x = r(\theta - \sin\theta), y = r(1 - \cos\theta)$ at the point $\theta = \pi/3$. At what points is the cycloid horizontal/vertical.

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$= \frac{\cancel{r}\sin\theta}{\cancel{r}(1-\cos\theta)}$$

$$= \frac{\sin\theta}{1-\cos\theta}$$

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = \frac{\sin\frac{\pi}{3}}{1-\cos\frac{\pi}{3}}$$

$$\frac{dx}{d\theta} = \frac{d}{d\theta} r(\theta - \sin\theta)$$

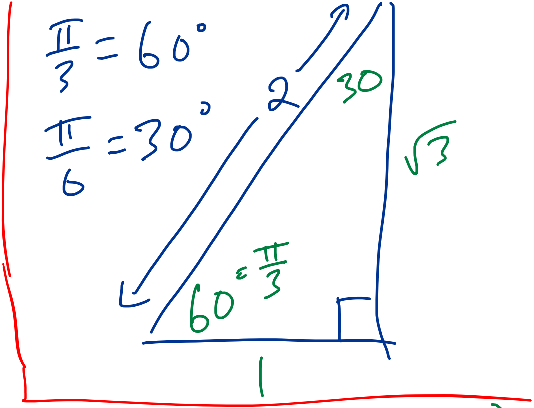
$$= r(1-\cos\theta)$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta} r(1-\cos\theta)$$

$$= r\sin\theta$$

$$= \frac{\frac{\sqrt{3}}{2}}{1-\frac{1}{2}}$$

$$= \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$



$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = \sqrt{3} \quad ,$$

$$x\left(\frac{\pi}{3}\right) = r\left(\frac{\pi}{3} - \sin\left(\frac{\pi}{3}\right)\right)$$

$$= r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)$$

$$y\left(\frac{\pi}{3}\right) = r\left(1 - \cos\left(\frac{\pi}{3}\right)\right)$$

$$= r\left(1 - \frac{1}{2}\right) = \frac{r}{2}$$

$$\frac{y - y\left(\frac{\pi}{3}\right)}{x - x\left(\frac{\pi}{3}\right)} = \sqrt{3}$$

$$y - y\left(\frac{\pi}{3}\right) = \sqrt{3}\left(x - x\left(\frac{\pi}{3}\right)\right)$$

$$y = \sqrt{3}\left(x - x\left(\frac{\pi}{3}\right)\right) + y\left(\frac{\pi}{3}\right)$$

$$y = \underbrace{\sqrt{3}}_{m}\, x + \underbrace{y\left(\frac{\pi}{3}\right) - \sqrt{3}\, x\left(\frac{\pi}{3}\right)}_{b}$$

$$y = \sqrt{3}\, x + \frac{r}{2} - \sqrt{3}\left(r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)\right)$$

$$= \sqrt{3}x + \frac{r}{2} - r\left(\frac{\pi}{\sqrt{3}} - \frac{3}{2}\right)$$

$$= \sqrt{3}x + \frac{r}{2} - \frac{r\pi}{\sqrt{3}} + \frac{3r}{2}$$

$$\boxed{Y = \sqrt{3}x + 2r - \frac{\pi}{\sqrt{3}}r}$$

$$\frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta}$$

Horizontal / Vertical

$\frac{dy}{dx} = 0$ $\qquad$ $\frac{dy}{dx} = \pm\infty$

$0 \le \theta < 2\pi$

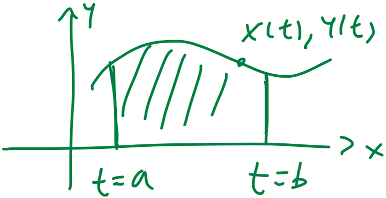
$\theta = 0, \pi$ $\quad$ $\sin\theta = 0$ or $\cos\theta = 1$

For $\theta = 0$, $\frac{dy}{dx}$ "=" $\frac{0}{0}$ $\quad$ Horiz. or Vert?

For $\theta = \pi$ $\quad \frac{dy}{dx} = \frac{\sin\pi}{1-\cos\pi} = \frac{0}{-2} = 0$

Horizontal

Area under parametric curves



$$(x(t), y(t)), \quad a \le t \le b$$

$$\text{Area} = \int_{x(a)}^{x(b)} y \, dx = \underbrace{\int_a^b y(t)\, x'(t)\, dt}_{\text{in terms of } t}$$

$$x = x(t)$$

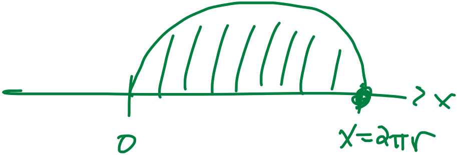
$$dx = x'(t)\, dt$$

$$= \int_a^b y \frac{dx}{dt} \, dt$$

$$x(2\pi) = r(2\pi - \sin(2\pi))$$

Find the area under one arch of the cycloid $x = r(\theta - \sin\theta), y = r(1 - \cos\theta)$.

$$\frac{dx}{d\theta} = r(1 - \cos\theta)$$



$$\text{Area} = \int_0^{2\pi} y \frac{dx}{d\theta}\, d\theta$$

$$= \int_0^{2\pi} r(1-\cos\theta)\, r(1-\cos\theta)\, d\theta$$

$$= \int_0^{2\pi} r^2(1-\cos\theta)^2\, d\theta$$

$$\boxed{\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)}$$

$$= \int_0^{2\pi} r^2(1 - 2\cos\theta + \cos^2\theta)\, d\theta$$

$$= \int_0^{2\pi} r^2 \left(1 - 2\cos\theta + \frac{1}{2}\left(1 + \cos(2\theta)\right)\right) d\theta$$

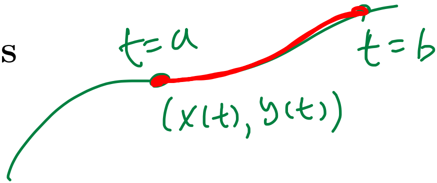
$$= \int_0^{2\pi} r^2 \left(1 - 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos(2\theta)\right) d\theta$$

$$= \int_0^{2\pi} r^2 \left(\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos(2\theta)\right) d\theta$$

$$= r^2 \left(\frac{3}{2}\theta - \cancel{2\sin\theta} + \cancel{\frac{1}{4}\sin(2\theta)}\right)\Bigg]_{\theta=0}^{\theta=2\pi}$$

$$= r^2 \left(\frac{3}{\cancel{2}} \cdot \cancel{2}\pi\right) = 3\pi r^2$$

Arclength of parametric curves



$$\text{Arclength} = \int_{x(a)}^{x(b)} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_a^b \sqrt{1+\left(\frac{y'(t)}{x'(t)}\right)^2}\; x'(t)\,dt$$

$$= \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\; dt$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t)}{x'(t)}$$

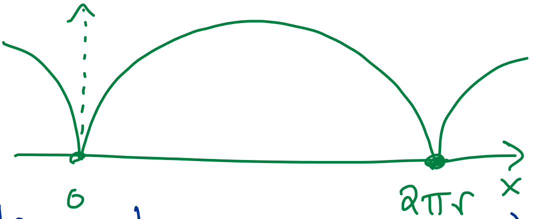
$$x = x(t)$$

$$dx = x'(t)\,dt$$

$$\text{Arclength} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$\theta = t$

Find the arclength of one arch of the cycloid $x = r(\theta - \sin\theta), y = r(1-\cos\theta)$.



One arch $\theta = 0 \to 2\pi$

$$\text{Arclength} = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$\frac{dx}{d\theta} = r(1-\cos\theta)$$

$$\frac{dy}{d\theta} = r\sin\theta$$

$$= \int_0^{2\pi} \sqrt{r^2(1-\cos\theta)^2 + r^2\sin^2\theta}\, d\theta$$

$$= \int_0^{2\pi} r\sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}\, d\theta$$

$$= \int_0^{2\pi} r\sqrt{2-2\cos\theta}\ d\theta$$

$$= \int_0^{2\pi} \sqrt{2}\, r\sqrt{1-\cos\theta}\ d\theta$$

$$= \int_0^{2\pi} \sqrt{2}\, r\sqrt{2\sin^2\left(\frac{\theta}{2}\right)}\ d\theta$$

$$\boxed{\begin{gathered} \sin^2 x = \frac{1}{2}(1-\cos 2x) \\ 2x = \theta\ ,\quad x = \frac{\theta}{2} \\ 2\sin^2\left(\frac{\theta}{2}\right) = 1-\cos\theta \end{gathered}}$$

$$= \int_0^{2\pi} 2r\sin\left(\frac{\theta}{2}\right) d\theta$$

$$u = \frac{\theta}{2}\ ,\quad du = \frac{d\theta}{2}\ ,\quad d\theta = 2du$$

$$u = 0 \longrightarrow \pi$$

$$= -4r\cos\left(\frac{\theta}{2}\right)\Bigg]_{\theta=0}^{\theta=2\pi}$$

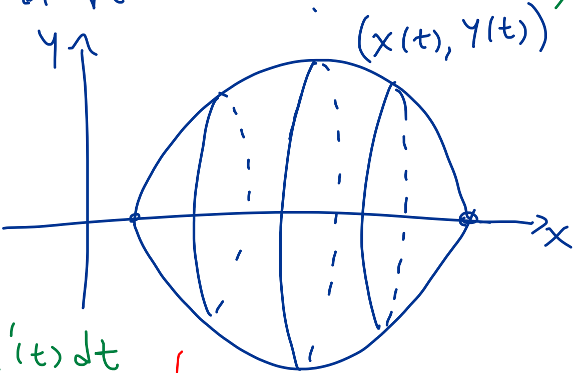
$$= \int_0^{\pi} 4r \sin(u)\, du = -4r\cos(u)\Big]_{u=0}^{u=\pi}$$

$$= -4r\left(\cos(\pi) - \cos(0)\right)$$

$$= -4r(-1-1) = 8r$$

Surface area (surface of revolution about x-axis)

$a \le t \le b$



Recall:

$$\text{Area} = \int_{x(a)}^{x(b)} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_a^b 2\pi y(t) \sqrt{1 + \left(\frac{y'(t)}{x'(t)}\right)^2}\, x'(t)\, dt$$

$$= \int_a^b 2\pi y(t) \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

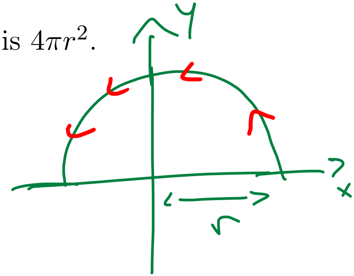
$$= \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$x = x(t)$

$dx = x'(t)\, dt$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t)}{x'(t)}$$

Show that the surface area of a sphere with radius r is $4\pi r^2$.



Rotate $x(\theta) = r\cos\theta$

$y(\theta) = r\sin\theta$

about x-axis.

$0 \le \theta \le \pi$

$$\text{Area} = \int_0^{\pi} 2\pi y \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{\pi} 2\pi r\sin\theta \sqrt{(-r\sin\theta)^2 + (r\cos\theta)^2}\, d\theta$$

$$\frac{dx}{d\theta} = -r\sin\theta$$

$$\frac{dy}{d\theta} = r\cos\theta$$

$$= \int_0^{\pi} 2\pi r \sin\theta \sqrt{r^2\sin^2\theta + r^2\cos^2\theta}\, d\theta$$

$$= \int_0^{\pi} 2\pi r^2 \sin\theta \underbrace{\sqrt{\sin^2\theta + \cos^2\theta}}_{=1}\, d\theta$$

$$= \int_0^{\pi} 2\pi r^2 \sin\theta \, d\theta$$

$$= 2\pi r^2 \left(-\cos\theta\right)\Big]_0^{\pi} = 2\pi r^2 \left(-\cos(\pi) + \cos(0)\right)$$

$$= 4\pi r^2$$

$$\frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta}, \quad \text{limit } \theta \to 0$$

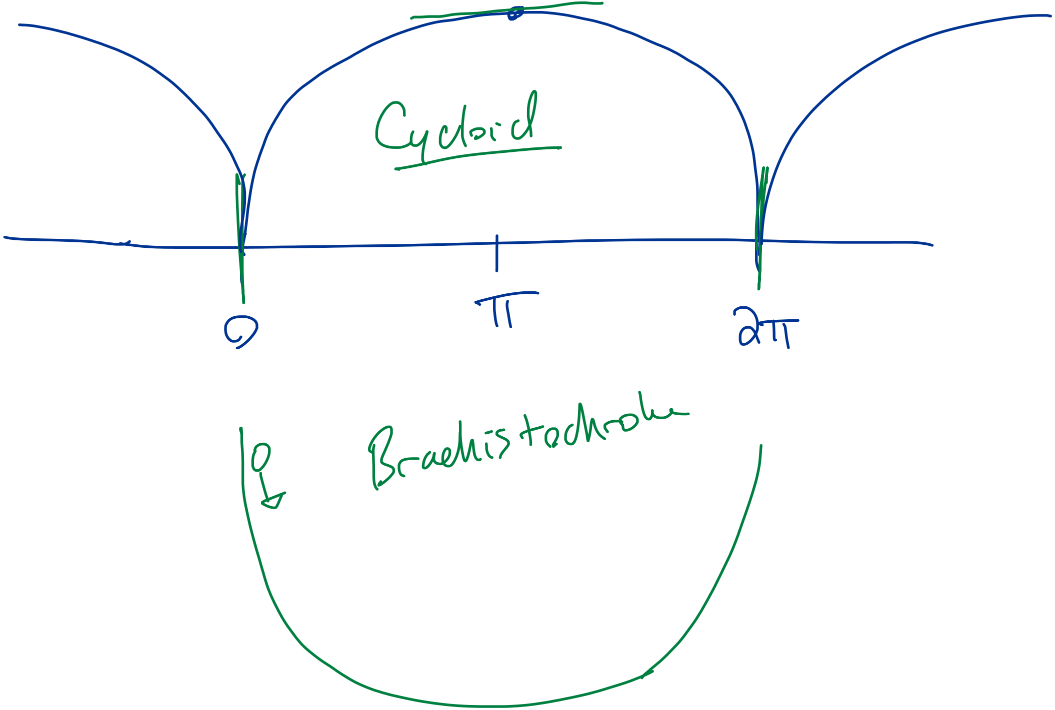
$$\lim_{\theta \to 0} \frac{\sin\theta}{1-\cos\theta} = \lim_{\theta \to 0} \frac{\frac{d}{d\theta}\sin\theta}{\frac{d}{d\theta}(1-\cos\theta)}$$

l'Hospital's rule

$$= \lim_{\theta \to 0} \frac{\cos\theta}{\sin\theta}$$

$$= \pm\infty \quad \begin{pmatrix} \theta \to 0^+ \\ \theta \to 0^- \end{pmatrix}$$

Vertical!


Cycloid
0
π
2π
Brachistochrone
g

Show that $x = \cos t$ $y = \sin t \cos t$ has 2 tangents at $(0,0)$. Sketch.

$t = \frac{\pi}{2}, \frac{3\pi}{2}$

Find tangent to

$x = 1+\sqrt{t}$, $y = e^{t^2}$ two ways $(2, e)$

$x = \sin \pi t$ $y = t^2 + t$ $(0, 2)$

Find y', y'': $x = \cos t$ $x = t - \ln t$

$y = \sin 2t$ $y = t + \ln t$

$0 < t < \pi$