Math 1272: Calculus II
10.3 Polar Coordinates

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Polar coordinates

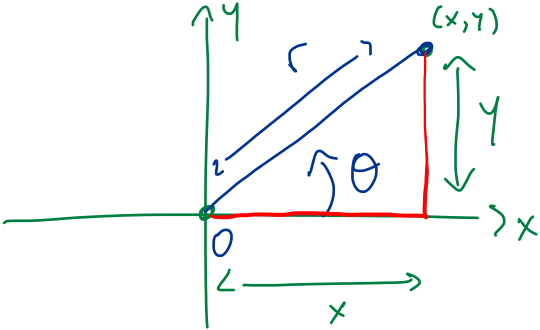
Every point (x, y) can be described by the distance from the origin

$$r = \sqrt{x^2 + y^2},$$

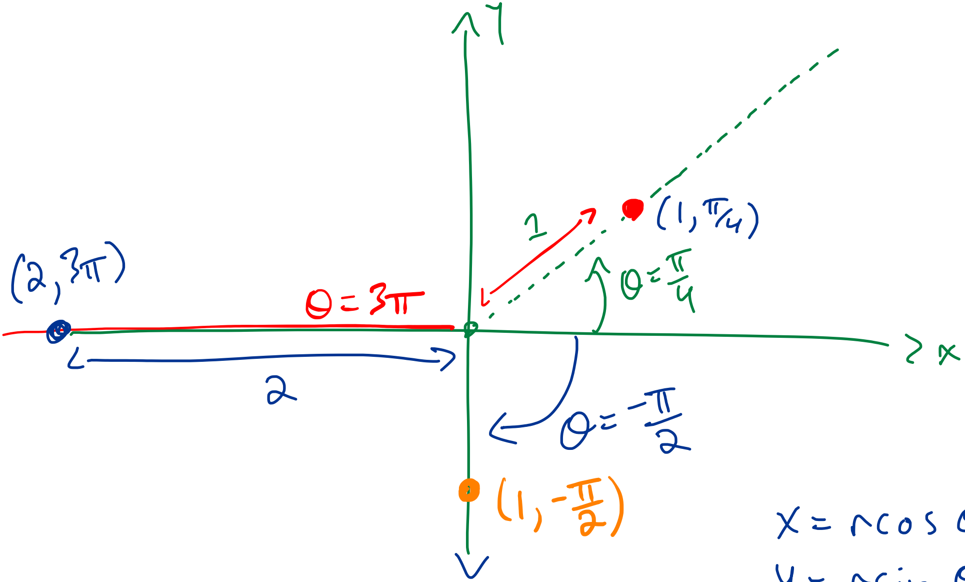
and the angle with the x-axis

$$\theta = \tan^{-1}\left(\frac{y}{x}\right).$$

The coordinates (r, θ) are called **polar coordinates**.



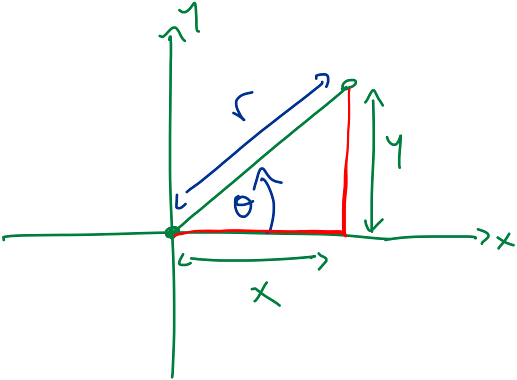
Plot the points $(1, \pi/4)$, $(2, 3\pi)$, and $(1, -\pi/2)$, given in polar coordinates.



$x = r\cos\theta$

$y = r\sin\theta$

For a point (r, θ) in polar coordinates, find the corresponding cartesian coordinates (x, y).



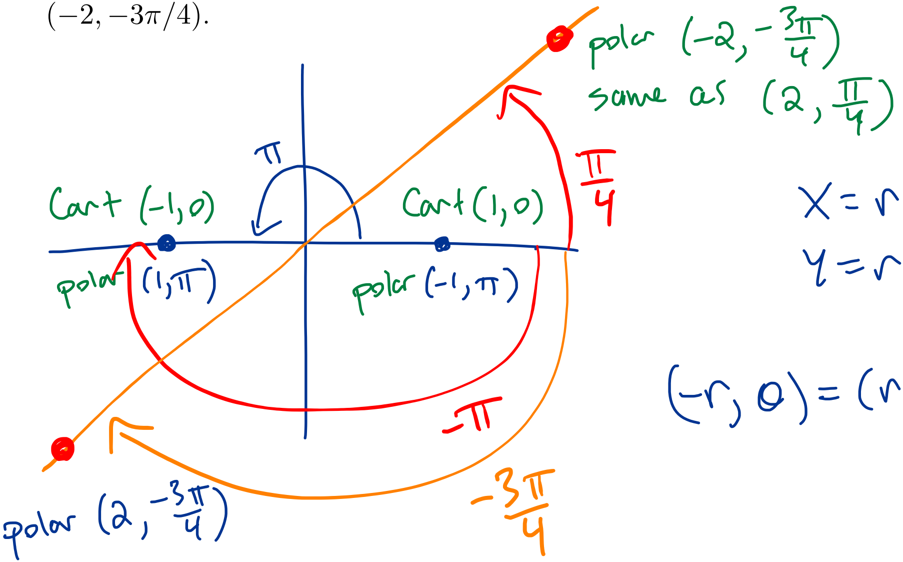
$$\cos\theta = \frac{x}{r}$$

$$\sin\theta = \frac{y}{r}$$

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

$\boxed{\text{Check}}$ $x^2 + y^2 = r^2$, $\tan\theta = \frac{y}{x}$

Note: We can allow $r < 0$ in polar coordinates. The points (r, θ) and $(-r, \theta)$ lie on the same ray with angle θ, but at anti-podal points. Plot $(-1, \pi)$ and $(-2, -3\pi/4)$.



$x = r\cos\theta$

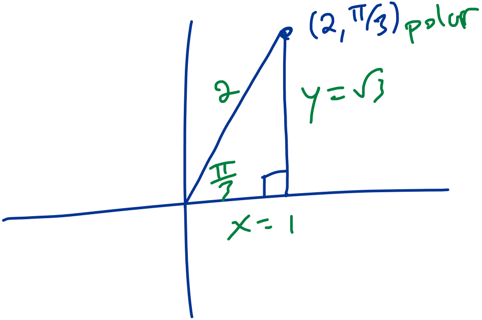
$y = r\sin\theta$

$(-r, \theta) = (r, \theta \pm \pi)$

Note: The polar coordinate point $(-r, \theta)$ is the same as $(r, \theta \pm \pi)$.

Convert the point $(2, \pi/3)$ from polar to cartesian coordinates.

$r = 2, \quad \theta = \pi/3$



$x = r\cos\theta$

$y = r\sin\theta$

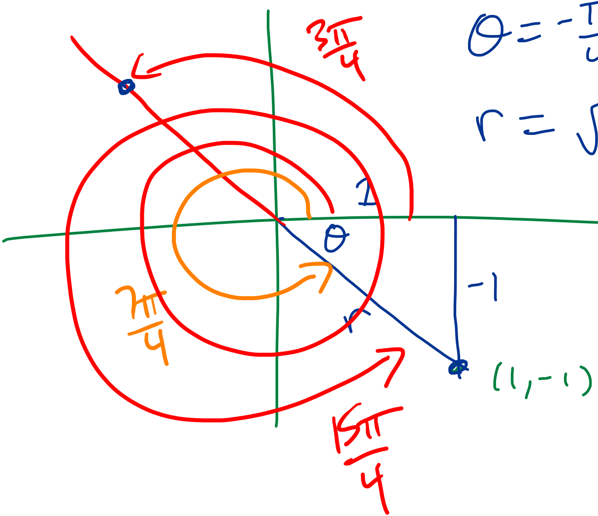
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$r = \sqrt{x^2 + y^2}$

$$x = 2\cos\left(\frac{\pi}{3}\right) = 2 \cdot \left(\frac{1}{2}\right) = 1$$

$$y = 2\sin\left(\frac{\pi}{3}\right) = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

Convert the point $(1, -1)$ from cartesian to polar coordinates.



$\theta = -\frac{\pi}{4}$

$r = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\left(\sqrt{2}, -\frac{\pi}{4}\right)$

$\left(-\sqrt{2}, \frac{3\pi}{4}\right)$

$\left(\sqrt{2}, \frac{7\pi}{4}\right)$

$\left(\sqrt{2}, \frac{15\pi}{4}\right)$

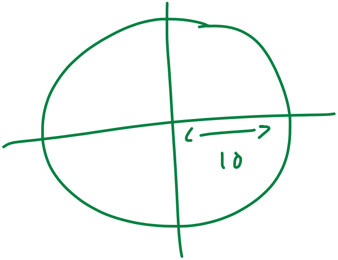
Polar curves

Curves can be represented in polar form as

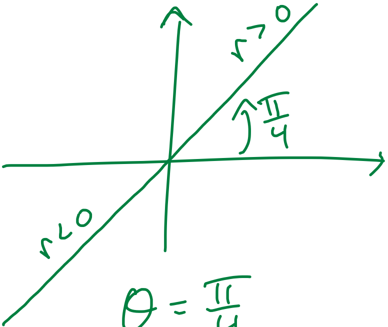
$$r = f(\theta).$$

The curve is the set of all (r, θ) satisfying $r = f(\theta)$.

Example: What curve is represented by the equation $r = 10$? How about $\theta = \pi/4$?



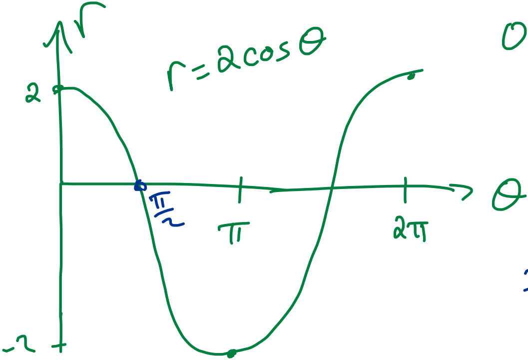
$r = 0$

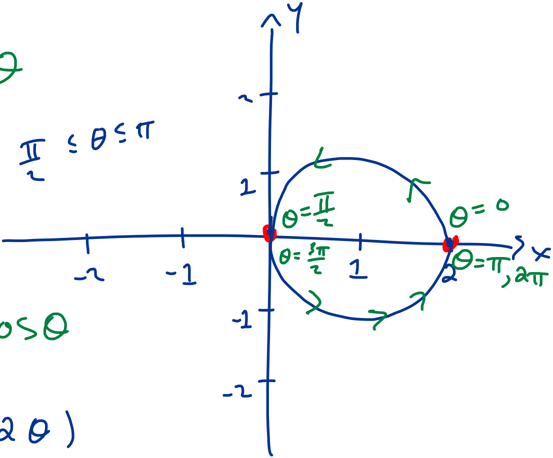


$\theta = \frac{\pi}{4}$

Sketch the curve $r = 2\cos\theta$ and convert to cartesian coordinates.

$0 \le \theta \le 2\pi$





$x = r\cos\theta$

$y = r\sin\theta \quad , \; r = 2\cos\theta$

$x = 2\cos^2\theta = 1 - \cos(2\theta)$

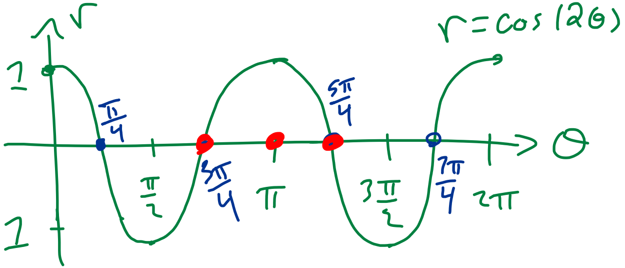
$y = 2\cos\theta\sin\theta = \sin(2\theta)$ ← trig identities

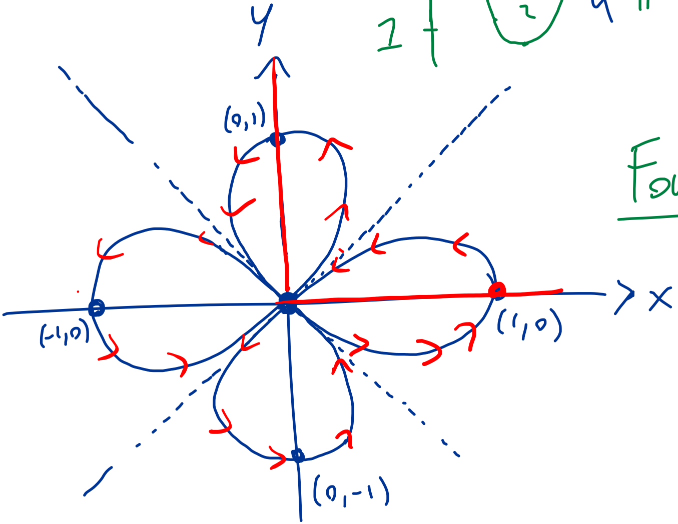
$$\left.\begin{aligned} x-1 &= -\cos(2\theta) \\ y &= \sin(2\theta) \end{aligned}\right\} \quad (x-1)^2+y^2 = \cos^2(2\theta) + \sin^2(2\theta) = 1$$

So $\boxed{(x-1)^2+y^2=1}$

2θ means circle traversed twice as $\theta: 0 \to 2\pi$

Sketch the curve $r = \cos(2\theta)$.





Four pedal rose

$r = \cos(2\theta)$ $r = f(\theta)$

$x = r\cos(\theta)$

$y = r\sin(\theta)$

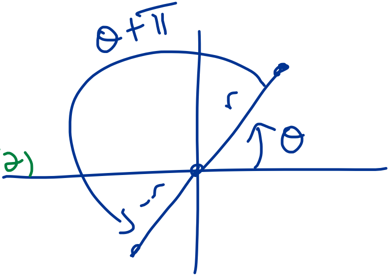
$$\boxed{\begin{aligned} x &= \cos(2\theta)\cos(\theta) \\ y &= \cos(2\theta)\sin(\theta) \end{aligned}}$$

The polar curve $F(r,\theta)=0$ is symmetric about the pole (origin) if either

$$F(r,\theta) = F(-r,\theta) \qquad (1)$$

or

$$F(r,\theta) = F(r,\theta+\pi) \qquad (2)$$



Ex: $r=10$ satisfies (2)

$r^2=100$ satisfies (1) and (2)

Both are circles

$F(r,\theta) = r - f(\theta)$

Symmetries in polar curves

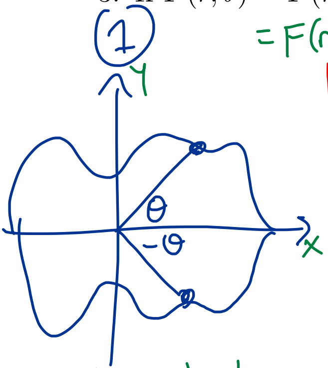
There are three important symmetries in a polar equation $\boxed{F(r,\theta) = 0.}$

1. If $F(r,\theta) = F(r,-\theta)$ then symmetric about $\theta = 0$.

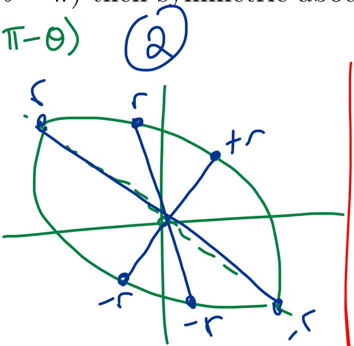
2. If $F(r,\theta) = F(-r,\theta)$ then symmetric about origin (the pole).

3. If $F(r,\theta) = \cancel{F(r,\theta-\pi)}$ then symmetric about axis $\theta = \pi/2$.

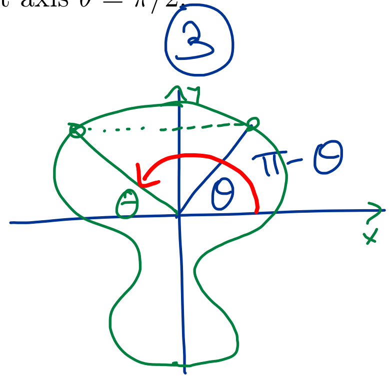
   $= F(r, \pi - \theta)$



symmetric about x-axis



Symmetric about origin



Symmetric about y-axis

Symmetries in polar curves

There are three important symmetries in a polar equation $F(r, \theta) = 0$.

1. If $F(r, \theta) = F(r, -\theta)$ then symmetric about $\theta = 0$.
2. If $F(r, \theta) = F(-r, \theta)$ then symmetric about origin (the pole).
3. If $F(r, \theta) = F(r, \cancel{\theta - \pi})$ then symmetric about axis $\theta = \pi/2$.

$\pi - \theta$

$F(r, \theta) = F(r, \theta + \pi)$

What are the symmetries in the last example $r = \cos(2\theta)$? Here

$$F(r, \theta) = r - \cos(2\theta).$$

1. yes since $\cos(-2\theta) = \cos(\theta)$.
2. NO, but does satisfy $F(r, \theta) = F(r, \theta + \pi)$
3. yes since $\cos(-2(\pi - \theta)) = \cos(-2\pi + 2\theta) = \cos(-2\theta)$.

Tangents to polar curves

$$r = f(\theta)$$

$$x = r\cos\theta = f(\theta)\cos\theta$$

$$y = r\sin\theta = f(\theta)\sin\theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$= \frac{\frac{d}{d\theta}\left(f(\theta)\sin\theta\right)}{\frac{d}{d\theta}\left(f(\theta)\cos\theta\right)}$$

$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

$f(\theta)$ $\quad f'(\theta) = \cos\theta$

$\shortparallel$

Find the slope of the tangent line to $r = 1 + \sin\theta$ when $\theta = \pi/3$. Find points where the tangent is horizontal/vertical.

$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

$$\frac{dy}{dx} = \left.\frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos\theta\cos\theta - (1+\sin\theta)\sin\theta}\right|_{\theta=\frac{\pi}{3}}$$

plug in $\theta = \frac{\pi}{3}$, $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

Horizontal or Vertical.

$$\frac{dy}{dx} = \frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos\theta\cos\theta - (1+\sin\theta)\sin\theta}$$

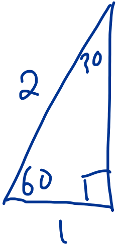
$$= \frac{\cos\theta\,(1+2\sin\theta)}{\cos^2\theta - \sin\theta - \sin^2\theta}$$

$$\frac{dy}{dx} = \frac{\cos\theta\,(1+2\sin\theta)}{1-\sin\theta-2\sin^2\theta}$$

$$\frac{dy}{dx} = \frac{\cos\theta\,(1+2\sin\theta)}{1-\sin\theta-2\sin^2\theta}$$

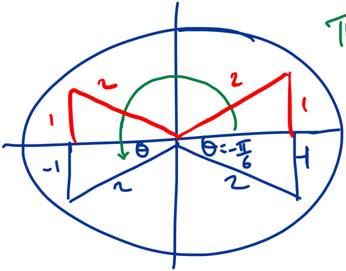
Numerator : $\cos\theta = 0$ or $\sin\theta = -\frac{1}{2}$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$



$30^\circ = \frac{\pi}{6}$

$\pi + \frac{\pi}{6} = \frac{7\pi}{6}$



$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

$\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$

$-\frac{\pi}{6} = \frac{11\pi}{6}$

Denominator:

$$1 - \sin\theta - 2\sin^2\theta = 0$$

$x = \sin\theta$

$$1 - x - 2x^2 = 0$$

$$2x^2 + x - 1 = 0$$

$$\sin\theta = x = \frac{-1 \pm \sqrt{1+8}}{4}$$

$$= -\frac{1}{4} \pm \frac{1}{4}\sqrt{9}$$

$$= -\frac{1}{4} \pm \frac{3}{4} = \frac{1}{2}, -1$$

$\sin\theta = \frac{1}{2}$ or $\sin\theta = -1$

$\sin\theta = \frac{1}{2}$ at $\theta = \frac{\pi}{6}$ and $\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

$\sin\theta = -1$ at $\theta = \frac{3\pi}{2}$

At $\theta = \frac{3\pi}{2}$, we have $\frac{dy}{dx} \cancel{=} \frac{0}{0}$

$$\lim_{\theta \to \frac{3\pi}{2}} \frac{dy}{dx} = \lim_{\theta \to \frac{3\pi}{2}} \frac{\cos\theta\,(1+2\sin\theta)}{1-\sin\theta-2\sin^2\theta}$$

l'Hospital

$$= \lim_{\theta \to \frac{3\pi}{2}} \frac{\cancel{\cos\theta\,(2\cos\theta)} - \sin\theta\,(1+2\sin\theta)}{-\cancel{\cos\theta} - 4\sin\cancel{\theta\cos}\theta}$$

$\cos\left(\frac{3\pi}{2}\right) = 0$

$\sin\left(\frac{3\pi}{2}\right) = -1$

$$\lim_{\theta \to \frac{3\pi}{2}} -\sin\theta\,(1+2\sin\theta) = -(-1)(1-2) = -1$$

$$= \pm\infty$$

<u>Horizontal</u> : $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$

<u>Vertical</u> : $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$