Math 1272: Calculus II
10.4 Areas and lengths in polar coordinates

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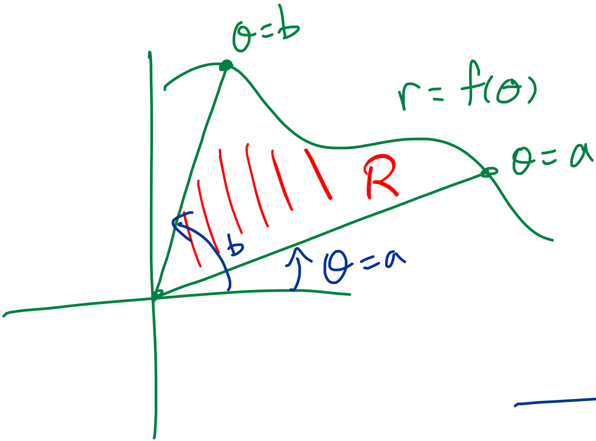
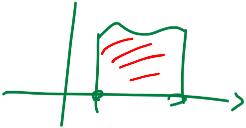
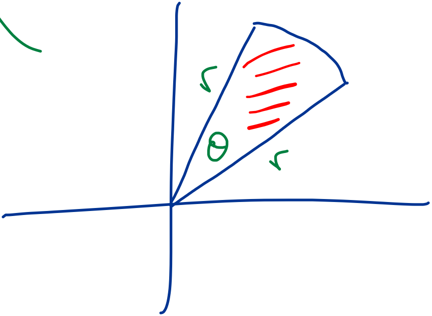
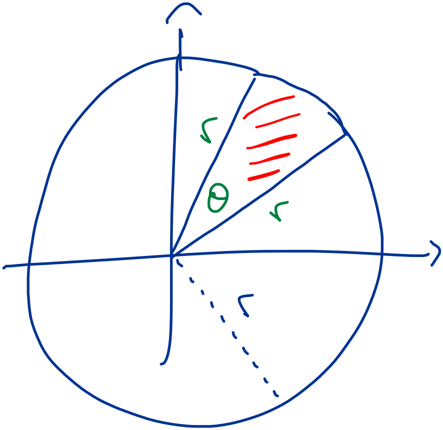
Area in polar coordinates

Let R be the region bounded by the polar function $r = f(\theta)$ for $a \leq \theta \leq b$, and the rays $\theta = a$ and $\theta = b$. What is the **area** of R?









Section of a circle

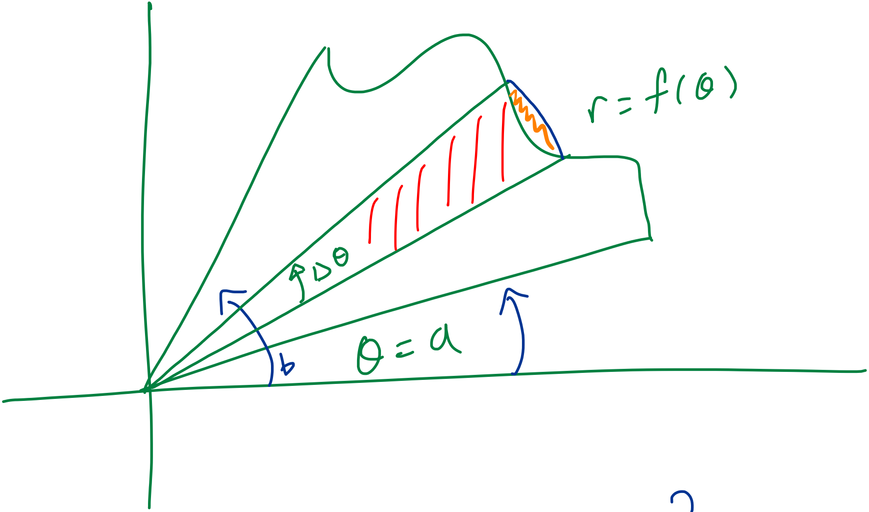
Whole circle has area

$$\pi r^2$$

Section is $\frac{\theta}{2\pi}$ fraction

of circle.

$$\text{Area of section} = \frac{\theta}{2\pi} \cdot \pi r^2 = \frac{\theta}{2} r^2$$



Area of a slice $\approx \frac{\Delta\theta}{2} \cdot r^2 = \frac{\Delta\theta}{2} f(\theta)^2$

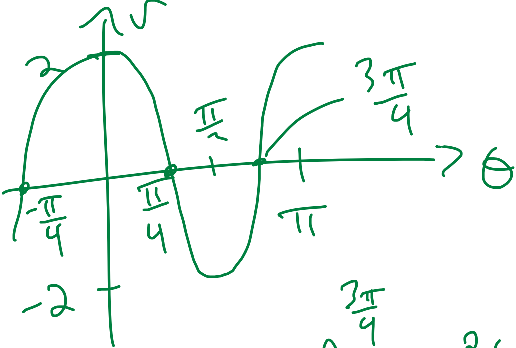
Break up $\theta = a \longrightarrow \theta = b$ into n

slices $\theta_0 = a$, $\theta_n = b$, $\theta_{i+1} - \theta_i = \Delta\theta$

$$\Delta\theta = \frac{b-a}{n}$$

$$\text{Area} \simeq \sum_{i=1}^{n} \frac{\Delta\theta}{2} f(\theta_i)^2 \simeq \boxed{\int_a^b \frac{f(\theta)^2}{2}\, d\theta}$$

Find the area enclosed by one loop of the four-leaved rose $r = \cos(2\theta)$.



$$\cos^2 x = \frac{1}{2}\left(1 + \cos(2x)\right)$$

$$x = 2\theta$$

$$2x = 4\theta$$

$$\text{Area} = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{\cos^2(2\theta)}{2}\, d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1}{2} \cdot \frac{1}{2}\left(1 + \cos(4\theta)\right) d\theta$$

$$= \frac{1}{4}\left[\theta + \frac{1}{4}\sin(4\theta)\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$$

$$= \frac{1}{4}\left(\frac{3\pi}{4} + \frac{1}{4}\overset{0}{\cancel{\sin(3\pi)}} - \frac{\pi}{4} - \frac{1}{4}\overset{0}{\cancel{\sin}}(\pi)\right)$$

$$= \frac{1}{4}\left(\frac{\pi}{2}\right) = \frac{\pi}{8}$$

$$2\sin\theta\cos\theta = \sin(2\theta)$$

$$2\sin^2\theta = 1 - \cos(2\theta)$$

Find the area of the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.

Circle:

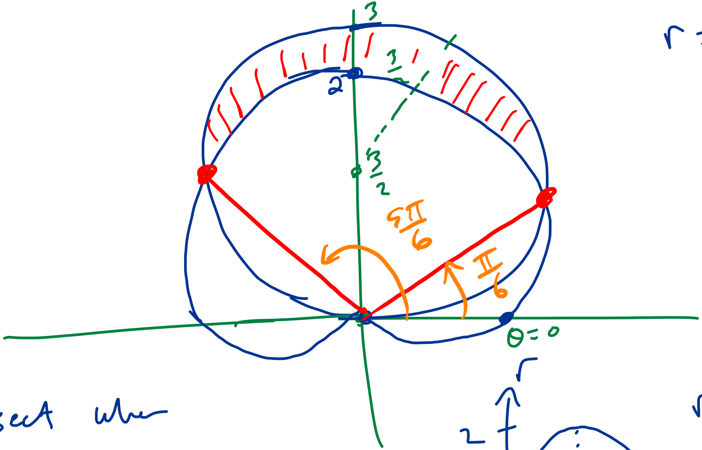
$$x = r\cos\theta = 3\sin\theta\cos\theta$$

$$= 3 \cdot \frac{1}{2}\sin(2\theta) = \frac{3}{2}\sin(2\theta)$$

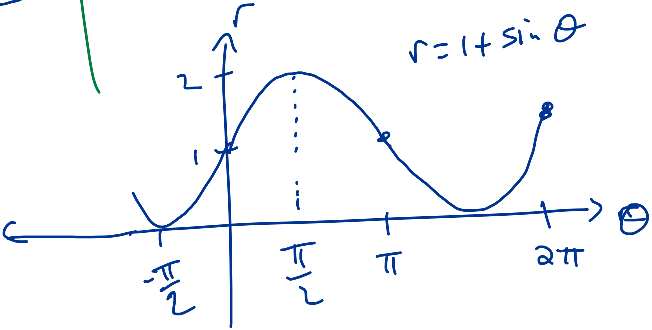
$$y = r\sin\theta = 3\sin^2\theta = \frac{3}{2}\left(1 - \cos(2\theta)\right).$$

$$= \frac{3}{2} - \frac{3}{2}\cos(2\theta)$$

$$x^2 + \left(y - \frac{3}{2}\right)^2 = \left(\frac{3}{2}\sin(2\theta)\right)^2 + \left(-\frac{3}{2}\cos(2\theta)\right)^2 = \left(\frac{3}{2}\right)^2$$



$r = 1 + \sin\theta$



Intersect when

$3\sin\theta = 1 + \sin\theta$

$2\sin\theta = 1$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

Area in circle $= \displaystyle\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{(3\sin\theta)^2}{2}\, d\theta$

Area in Cardiod $= \displaystyle\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{(1+\sin\theta)^2}{2}\, d\theta$

Area between $= \displaystyle\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}\left((3\sin\theta)^2 - (1+\sin\theta)^2\right) d\theta$

$$= \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}\left(9\sin^2\theta - 1 - 2\sin\theta - \sin^2\theta\right) d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}\left(8\sin^2\theta - 1 - 2\sin\theta\right)d\theta$$

Area

$$= \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}\left(4 - 4\cos(2\theta) - 1 - 2\sin\theta\right)d\theta$$

Integrate!

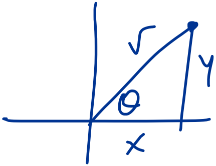
Arclength in polar coordinates

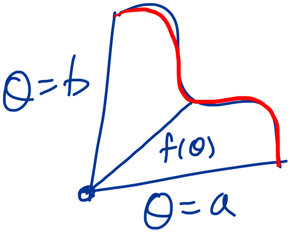
Consider the curve given by $r = f(\theta)$ for $a \le \theta \le b$. What is the **arclength**?

$x = r\cos\theta$

$y = r\sin\theta$

$r = f(\theta)$





$$\left.\begin{aligned} x &= f(\theta)\cos\theta \\ y &= f(\theta)\sin\theta \end{aligned}\right\}$$ Parametric equations.

Arclength: $\displaystyle\int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$

$$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta \qquad \text{(product rule)}$$

$$\left(\frac{dx}{d\theta}\right)^2 = \left(f'(\theta)\cos\theta - f(\theta)\sin\theta\right)^2$$

$$= (f')^2\cos^2\theta + f^2\sin^2\theta - \cancel{2f'f\cos\theta\sin\theta}$$

$$\left(\frac{dy}{d\theta}\right)^2 = \left(f'(\theta)\sin\theta + f(\theta)\cos\theta\right)^2$$

$$= (f')^2\sin^2\theta + f^2\cos^2\theta + \cancel{2f'f\cos\theta\sin\theta}$$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = (f')^2\cos^2\theta + f^2\sin^2\theta + (f')^2\sin^2\theta + f^2\cos^2\theta$$

$$= f'(\theta)^2\underbrace{(\cos^2\theta + \sin^2\theta)}_{=1} + f(\theta)^2\underbrace{(\cos^2\theta + \sin^2\theta)}_{=1}$$

$$= f'(\theta)^2 + f(\theta)^2$$

Arclength $= \int_a^b \sqrt{f'(\theta)^2 + f(\theta)^2}\, d\theta$

$$\boxed{= \int_a^b \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta}$$

$r = f(\theta)$

$\frac{dr}{d\theta} = f'(\theta)$

Find the arclength of the cardioid $r = 1 + \sin\theta$.

$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$r^2 = (1+\sin\theta)^2$$

$$= 1 + \sin^2\theta + 2\sin\theta$$

$$\left(\frac{dr}{d\theta}\right)^2 = (\cos\theta)^2$$

$$= \cos^2\theta$$

$$= \int_0^{2\pi} \sqrt{1 + \sin^2\theta + 2\sin\theta + \cos^2\theta}\, d\theta$$

$=1$

$$= \int_0^{2\pi} \sqrt{2 + 2\sin\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{2}\sqrt{1+\sin\theta}\, d\theta$$

$$= \sqrt{2} \int_0^{2\pi} \frac{\sqrt{1+\sin\theta}\ \sqrt{1-\sin\theta}}{\sqrt{1-\sin\theta}}\, d\theta$$

$$= \sqrt{2} \int_0^{2\pi} \frac{\sqrt{1-\sin^2\theta}}{\sqrt{1-\sin\theta}}\, d\theta$$

$$(1+\sin\theta)(1-\sin\theta)$$

$$= 1-\sin^2\theta$$

$$= \sqrt{2} \int_0^{2\pi} \frac{\sqrt{\cos^2\theta}}{\sqrt{1-\sin\theta}}\, d\theta$$

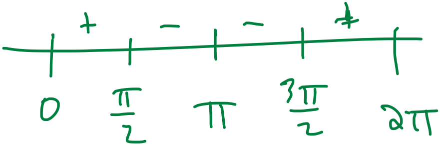
$$\sqrt{(-1)^2} = 1$$

$$= \sqrt{2} \int_0^{2\pi} \frac{|\cos\theta|}{\sqrt{1-\sin\theta}}\, d\theta$$

$$\sqrt{x^2} = |x|$$

$$= \sqrt{2}\int_0^{\pi/2} \frac{\cos\theta}{\sqrt{1-\sin\theta}}\,d\theta \;-\; \sqrt{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\cos\theta}{\sqrt{1-\sin\theta}}\,d\theta$$

$$+ \int_{\frac{3\pi}{2}}^{2\pi} \frac{\cos\theta}{\sqrt{1-\sin\theta}}\,d\theta$$

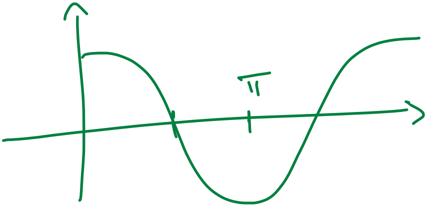


Substitution

$$u = 1 - \sin\theta$$

$$du = -\cos\theta\, d\theta$$

$$\frac{\cos\theta\, d\theta}{\sqrt{1-\sin\theta}} = \frac{-du}{\sqrt{u}}$$



Exercise to integrate