Math 1272: Calculus II
11.1 Sequences

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Sequences

A **sequence** is a (possibly infinite) list of numbers

$$a_1, a_2, a_2, \ldots, a_n, \ldots$$

Notation: $\{a_1, a_2, a_3, \ldots\}$, $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$.

Examples

1. $\left\{\frac{1}{n}\right\}_{n=1}^{\infty} = \{1, 1/2, 1/3, \ldots\}$.
2. $\{p_n\}$ where p_n is the population of the world on Jan 1 in year n
3. Fibonacci sequence

$$\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}.$$

Limits

A sequence $\{a_n\}_{n=1}^{\infty}$ has a **limit** L, which we write as

$$\lim_{n\to\infty} a_n = L$$

or $a_n \to L$ as $n \to \infty$, if we can make the terms a_n as close to L as we like by taking n large.

Examples

1. The sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ has limit $L = 0$ as $n \to \infty$ (**converges**)
2. Fibonacci sequence

   $$\{1, 1, 2, 3, 5, 8, 13, 21, \dots\}$$

   does not have a limit as $n \to \infty$. (**diverges**)
3. What about $\{1, -1, 1, -1, 1, -1, 1, -1, \dots\}$? (**converges/diverges?**)

Definition of Limit

We say $\lim_{n \to \infty} a_n = L$ if for every

$\varepsilon > 0$ there exists $N \geq 1$

such that

$$|a_n - L| < \varepsilon$$

for all $n \geq N$.

Diverging to ∞

While the Fibonacci sequence

$$\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$$

is divergent, it increases monotonically without bound. Such sequences diverge to ∞:

Definition: We say $\lim_{n\to\infty} a_n = \infty$ if for every large $M > 0$ there is an integer N such that

$$\text{if } n > N \text{ then } a_n > M.$$

Rules for limits

If $\{a_n\}$ and $\{b_n\}$ are convergent then

$a_n = \{1, -1, 1, -1, 1, -1, \ldots\}$

$b_n = \{-1, 1, -1, 1, -1, 1, \ldots\}$

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} c a_n = c \lim_{n\to\infty} a_n$$

$$\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

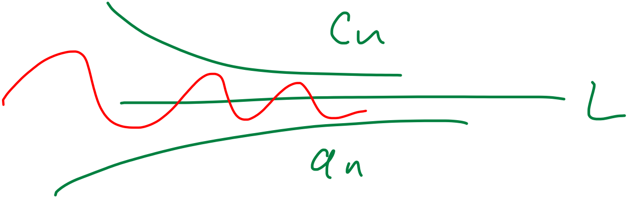
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if} \quad \lim_{n\to\infty} b_n \neq 0$$

$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \quad \text{if } p > 0 \text{ and } a_n > 0.$$

Useful theorems

Squeeze theorem:

If $a_n \leq b_n \leq c_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.



Absolute values:

If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Find $\lim_{n\to\infty} \frac{n}{n-\pi}$.

$$\lim_{n\to\infty} \frac{n}{n-\pi} \, \frac{\frac{1}{n}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{1}{1-\frac{\pi}{n}} \qquad \frac{\pi}{n} \to 0$$

$$= \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} \left(1-\frac{\pi}{n}\right)} = \frac{1}{1} = 1$$

Is the sequence $a_n = \frac{n^2}{n+1}$ convergent or divergent?

1 Note $a_n = \frac{n^2}{n+1} \geq \frac{n^2}{n+n} = \frac{n^2}{2n} = \frac{n}{2}$

Since $\lim_{n\to\infty} n = \infty$, $\lim_{n\to\infty} a_n = \infty$

2 $a_n = \frac{n}{1+\frac{1}{n}}$

$\lim_{n\to\infty} n = \infty$

$\lim_{n\to\infty} \left(1+\frac{1}{n}\right) = 1$

Evaluate $\lim_{n\to\infty} \frac{(-1)^n}{n}$ if it exists. , $a_n = \frac{(-1)^n}{n}$

$-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}, \dots$

$|a_n| = \frac{1}{n}$ and $\lim_{n\to\infty} \frac{1}{n} = 0$

$\lim_{n\to\infty} |a_n| = 0 \implies \lim_{n\to\infty} a_n = 0$

Calculate $\lim_{n\to\infty} \frac{\ln n}{n}$.

$\ln(n^{1/n})$

l'Hospital

If $\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1}$

$= \lim_{x\to\infty} \frac{1}{x} = 0$

Difference $n = 1, 2, 3, 4, \ldots$

x any number.

Last example used the following fact:

If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ for n an integer, then $\lim_{n\to\infty} a_n = L$.

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↳ Replace n by x and use l'Hospital

Another useful fact:

If $\lim_{n\to\infty} a_n = L$ and $f(x)$ is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(L).$$

Limits commute with continuous functions

Find $\lim_{n\to\infty} \sin(\pi/n)$.

$a_n = \frac{\pi}{n}$, $\lim_{n\to\infty} a_n = 0$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\,\, \parallel$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\, L$

$\sin(x)$ is continuous at $L = 0$

So

$$\lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right) = \sin\left(\lim_{n\to\infty} \frac{\pi}{n}\right)$$

$$= \sin(0) = 0$$

For what values of r is $\{r^n\}$ convergent? $r > 0$

$$\{r^n\}_{n=1}^{\infty} = \{r, r^2, r^3, r^4, r^5, \ldots\}$$

$$r^n = e^{\ln(r^n)} = e^{n\ln(r)} = e^{kn}, \quad k = \ln(r)$$

What is $\lim_{n\to\infty} e^{kn}$?

$$\frac{dP}{dt} = kP$$

$$n = t$$

∞ if $k > 0$

0 if $k < 0$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\ln(r) > 0 \qquad\qquad \ln(r) < 0$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$r > 1 \qquad\qquad 0 < r < 1$$

$$\lim_{n\to\infty} r^n = 0 \quad \text{if} \quad 0 < r < 1$$

$$\lim_{n\to\infty} r^n = \infty \quad \text{if} \quad r > 1.$$

$$\lim_{n\to\infty} r^n = 1 \quad \text{if} \quad r = 1$$

$\lim_{n \to \infty} r^n = 0$ if $|r| < 1$.

If $r = -1$, diverges.

If $r < -1$, diverges.

Convergent for $-1 < r \leq 1$

Monotonic sequences

A sequence is **increasing** if

$$a_1 < a_2 < a_3 < a_4 < a_5 < \cdots$$

and **decreasing** if

$$a_1 > a_2 > a_3 > a_4 > a_5 > \cdots$$

A sequence is **monotonic** if it is either increasing or decreasing.

Example: $\{\frac{2}{2+n}\}$ is decreasing.

Bounded sequences

A sequence is **bounded above** if there is some number M so that

$$a_n \leq M \text{ for all } n \geq 1.$$

A sequence is **bounded below** if there is some number m so that

$$a_n \geq m \text{ for all } n \geq 1.$$

A sequence is **bounded** if it is bounded above and below.

Fact: Every **monotonic** and **bounded** sequence is convergent.

Consider the sequence defined by the recurrance relation

$$a_1 = 2 \text{ and } a_{n+1} = \frac{1 + a_n}{2}.$$

Show that the sequence $\{a_n\}$ is convergent. What is the limit?

If $a_n \le 2$ then $a_{n+1} = \frac{1+a_n}{2} \le \frac{1+2}{2} = \frac{3}{2} \le 2$

induction $\Rightarrow$ $a_n \le 2$ for all n.

If $a_n \ge 1$ then $a_{n+1} = \frac{1+a_n}{2} \ge \frac{1+1}{2} = 1$

induction $\rightarrow$ $a_n \ge 1$

So $1 \le a_n \le 2$ for all n

Hence $\{a_n\}_{n=1}^{\infty}$ is bounded.

Claim: a_n decreasing.

$$a_2 = \frac{1+a_1}{2} = \frac{1+2}{2} = \frac{3}{2} \leq 2 = a_1$$

If $a_{n-1} \geq a_n$ then

$$a_{n+1} = \frac{1+a_n}{2} \leq \frac{1+a_{n-1}}{2} = a_n$$

induction $\Rightarrow$ a_n decreasing.

$\{a_n\}_{n=1}^{\infty}$ is decreasing and bounded, hence it converges. Write

$$L = \lim_{n\to\infty} a_n$$

$$= \lim_{n\to\infty} \frac{1+a_{n-1}}{2}$$

$$= \frac{1+\lim_{n\to\infty} a_{n-1}}{2} = \frac{1+L}{2}$$

$$L = \frac{1+L}{2}$$

$$2L = 1 + L \implies \boxed{L = 1}$$