Math 1272: Calculus II
11.2 Series

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Series

A **series** is a sum of a sequence $\{a_n\}$ of numbers

$$a_1 + a_2 + a_3 + a_4 + \cdots + a_n + \cdots$$

Notation: $\sum a_n$ or $\sum_{n=1}^{\infty} a_n$.

Example: The number $\pi = 3.1415926535\ldots$ is actually the series

$$3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5} + \frac{2}{10^6} + \frac{6}{10^7} + \frac{5}{10^8} + \cdots$$

Question: Does it make sense to consider an infinite sum

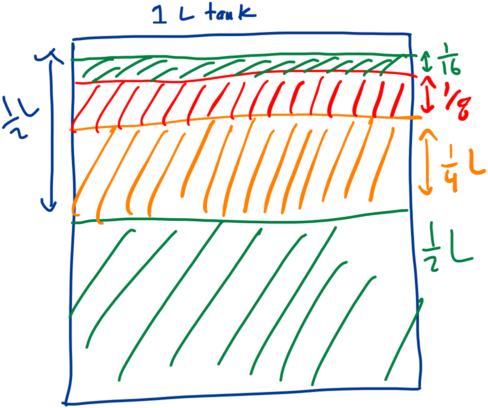
$$a_1 + a_2 + a_3 + a_4 + \cdots + a_n + \cdots ?$$

If $a_n > 0$ for all n, need the sum be ∞?

Consider the series

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots \overset{?}{=} 1$$

- Each term is $\frac{1}{2}$ of previous.



Step (n)	amount added	amount left to fill
1	1/2	1/2
2	1/4	1/4
3	1/8	1/8
4	1/16	1/16

Convergence

Define the partial sums

$$s_n = \sum_{i=1}^{n} a_i = a_1 + \cdots + a_n.$$

If the seqeunce $\{s_n\}$ is convergenet and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum_{n=1}^{\infty} a_n$ is **convergent** and

$$\sum_{n=1}^{\infty} a_n = s \qquad \left(\text{or } \sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i \right).$$

The number s is the **sum** of the series. If the sequence $\{s_n\}$ is divergent, then the series is divergent.

Geometric series

An important series is the geometric series

$$a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots = \sum_{n=1}^{\infty} ar^{n-1}.$$

Previous example $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$

is geometric with $a = 1$, $r = \frac{1}{2}$

Question: Does geometric series converge?

What does it converge to?

Partial Sums

$$S_n = a + \cancel{ar} + \cancel{ar^2} + \cdots + \cancel{ar^n}$$

Trick:

$$rS_n = \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \cdots + \cancel{ar^n} + ar^{n+1}$$

$$rS_n - S_n = ar^{n+1} - a$$

$$(r-1)S_n = ar^{n+1} - a \qquad (r \neq 1)$$

$$\boxed{S_n = \frac{ar^{n+1} - a}{r-1}}$$

Question: For what values of r does $\{S_n\}$ converge as $n \to \infty$?

If $|r| < 1$, then $\lim_{n\to\infty} r^{n+1} = 0$ and

$$\sum_{n=0}^{\infty} ar^n = \lim_{n\to\infty} S_n = \frac{a}{1-r} \qquad \text{(converges)}$$

If $|r| > 1$, then $\{r^n\}$ diverges, so the series diverges.

If $r = \pm 1$:

$r = 1$: $a + a + a + a + \cdots$ (diverges for $a \neq 0$)

If $r = -1$: $a - a + a - a + a - a + \cdots$

$$S_n = \frac{ar^{n+1} - a}{r-1} = \frac{a(-1)^{n+1} - a}{-2}$$

(diverges).
$$= \frac{a\left((-1)^{n+1} - 1\right)}{2}$$

Geometric series

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots + ar^n + \cdots$$

converges to $\boxed{\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}}$ if $|r|<1$

and diverges otherwise $(|r| \geq 1)$.

Find the sum of the geometric series

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$$

$$a + ar + ar^2 + ar^3 + \cdots \qquad a = 5$$

$$= a(1 + r + r^2 + r^3 + \cdots)$$

$$5\left(1 \underbrace{- \frac{2}{3}}_{=r} + \underbrace{\frac{4}{9}}_{=r^2} \underbrace{- \frac{8}{27}}_{=r^3} + \cdots\right) \qquad r = -\frac{2}{3}$$

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots = \frac{a}{1-r} = \frac{5}{1+\frac{2}{3}}$$

$$= \frac{5}{\frac{5}{3}} = 5 \cdot \frac{3}{5} = 3$$

In the geometric series $\sum r^{n\cancel{-1}}$, the series is convergent when $|r| < 1$ so that

$$\lim_{n\to\infty} r^n = 0.$$

Question: If $\lim_{n\to\infty} a_n = 0$, must $\sum_{n=1}^{\infty} a_n$ converge?

Example: Examine the convergence/divergence of the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots .$$

Claim: Harmonic series diverges.

$$1 + \underbrace{\frac{1}{2}}_{\frac{1}{2}} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq 2 \cdot \frac{1}{4} = \frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq 4 \cdot \frac{1}{8} = \frac{1}{2}} + \underbrace{\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}}_{\geq 8 \cdot \frac{1}{16} = \frac{1}{2}}$$

$$+ \underbrace{\frac{1}{17} + \frac{1}{18} + \cdots + \frac{1}{32}}_{\geq 16 \cdot \frac{1}{32} = \frac{1}{2}} + \cdots$$

diverges

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

Tests for convergence/divergence

1. If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Proof: $S_n = a_1 + a_2 + \cdots + a_n$

$S_{n-1} = a_1 + a_2 + \cdots + a_{n-1}$

$S_n - S_{n-1} = a_n$

$0 = \lim_{n\to\infty} S_n - S_{n-1} = \lim_{n\to\infty} a_n$

Contrapositive

2. ~~Converse~~ If $\lim_{n\to\infty} a_n \neq 0$ (or a_n does not converge), then $\sum a_n$ is divergent.

Show that the series $\sum_{n=1}^{\infty} \frac{n^2}{4n^2-2}$ diverges.

$$a_n = \frac{n^2}{4n^2-2}$$

$$a_n = \frac{n^2}{4n^2-2} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \frac{1}{4-\cancelto{0}{\frac{2}{n^2}}}$$

$$\lim_{n\to\infty} a_n = \frac{1}{4} \neq 0$$

Since $\lim_{n\to\infty} a_n \neq 0$, series diverges

$$\lim_{x\to\infty} \frac{x^2}{4x^2-2} \overset{\text{l'Hospital}}{=} \lim_{x\to\infty} \frac{2\cancel{x}}{8\cancel{x}} = \frac{1}{4}$$

Properties of convergent series

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent then

$$\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$$

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

$$\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$$

Example: Evaluate $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ $\frac{1}{n^2+n} = \frac{1}{n(n+1)}$

Partial: $\frac{1}{\boxed{n(n+1)}} = \frac{A}{n} + \frac{B}{n+1} = \frac{1}{n} - \frac{1}{n+1}$

$$1 = A(n+1) + Bn$$

$$0 \cdot n + 1 = \underbrace{(A+B)}_{=0} n + \underset{\overset{\|}{1}}{A}$$

$A = 1$

$B = -1$

$$\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{6}}\right) + \left(\cancel{\frac{1}{6}} - \frac{1}{7}\right) + \cdots$$

Telescoping sums.

$$S_n = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1} \right) = 1 - \frac{1}{n+1}$$