Math 1272: Calculus II
11.3 The integral test

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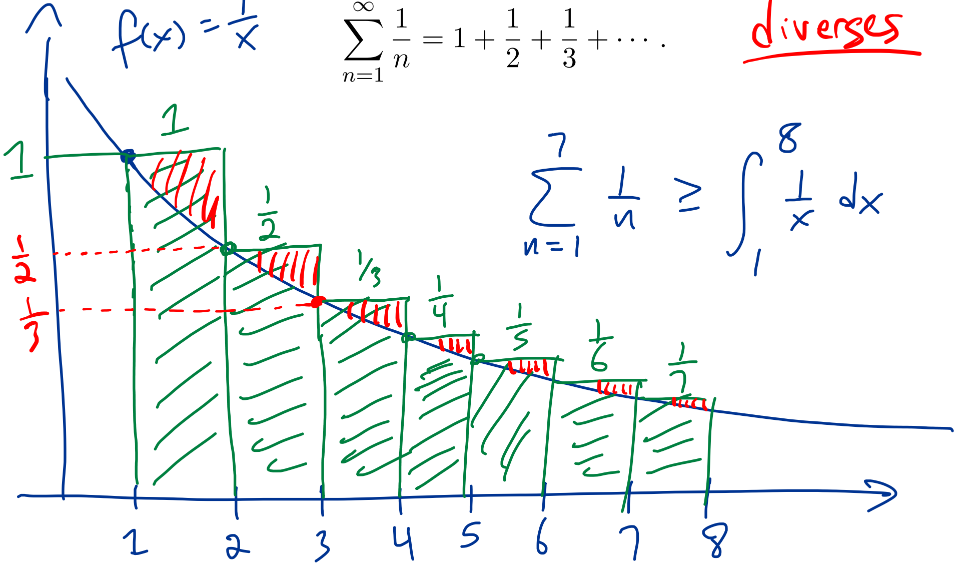
Harmonic series

Let us return to the harmonic series

$f(x) = \frac{1}{x}$

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots .$$

diverges



$$\sum_{n=1}^{7} \frac{1}{n} \geq \int_1^8 \frac{1}{x}\,dx$$

In general

$$S_n = \sum_{i=1}^{n} \frac{1}{i} \geq \int_1^{n+1} \frac{1}{x}\,dx$$

$$= \ln(x)\Big|_1^{n+1}$$

$$= \ln(n+1) \xrightarrow[\infty]{n} \infty$$

Hence $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Let's consider a general series

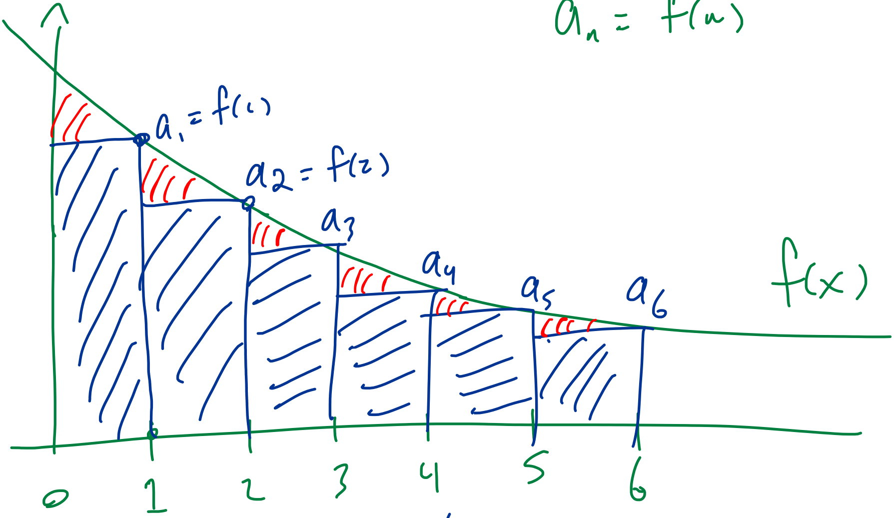
$$\sum_{n=1}^{\infty} a_n \quad \text{where } a_n = f(n)$$

and f is **positive** an **decreasing**.

By above argument

$$\boxed{S_n = \sum_{i=1}^{n} a_i \geq \int_1^{n+1} f(x)\,dx}$$

$a_n = f(n)$



$$\sum_{i=1}^{6} a_i \le \int_0^6 f(x)\,dx$$

In general $$\boxed{S_n = \sum_{i=1}^{n} a_i \leq \int_0^n f(x)\,dx}$$

$$\boxed{\int_1^{n+1} f(x)\,dx \leq S_n \leq \int_0^n f(x)\,dx}$$

Integral test

$n > N$

If f is continuous, positive, and decreasing, and $a_n = f(n)$, then $\sum a_n$ is convergent if and only if $\int_1^\infty f(x)\,dx$ is convergent.

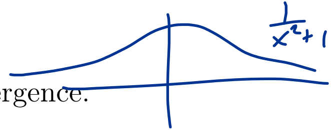
- If $\int_1^\infty f(x)\,dx$ is convergent then $\sum_{n=1}^\infty a_n$ is convergent.
- If $\int_1^\infty f(x)\,dx$ is divergent then $\sum_{n=1}^\infty a_n$ is divergent.

$\int_N^\infty f(x)\,dx$

Warning: In general, the integral and sum do not give the same value, so

$$\int_1^\infty f(x)\,dx \neq \sum_{n=1}^\infty a_n.$$

Test the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence or divergence.



$f(x) = \frac{1}{x^2+1}$. Integral test

$$\int_1^\infty \frac{1}{x^2+1}\,dx = \lim_{T\to\infty} \int_1^T \frac{1}{x^2+1}\,dx$$

$$= \lim_{T\to\infty} \arctan(x)\Big]_1^T$$

$\arctan = \tan^{-1}$

$$= \lim_{T\to\infty} \arctan(T) - \arctan(1)$$

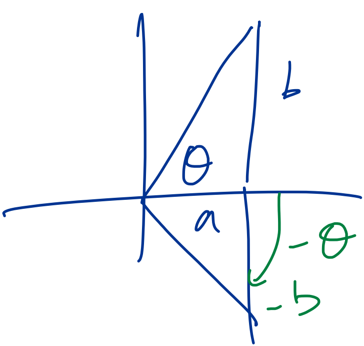
$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

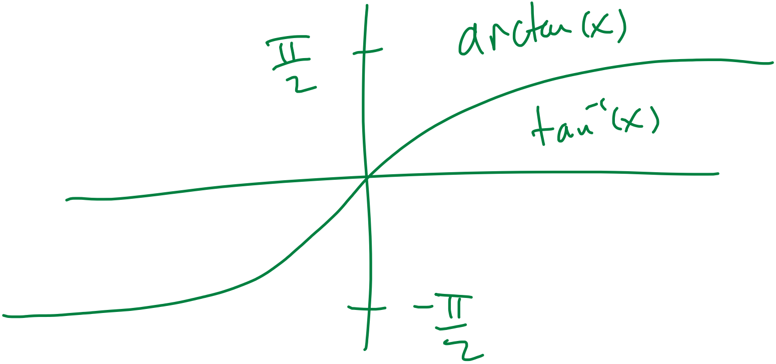
$$\tan\theta = \frac{b}{a}$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

$$\theta = \tan^{-1}(x)$$

$$x = \frac{b}{a}$$





$f(x) = \frac{1}{x^p}$

For what values of p does $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge?

By integral test, compare with $\int_1^\infty \frac{1}{x^p}\,dx$

$\int_1^\infty \frac{1}{x^p}\,dx$ converges for $p > 1$ (p-test)

diverges for $p \leq 1$

Same is true for $\sum \frac{1}{n^p}$.

($p = 1$) Harmonic series $\sum \frac{1}{n}$ diverges.

Estimating the sum of a series

Suppose we have a convergent series

$$s = \sum_{n=1}^{\infty} a_n. = a_1 + a_2 + \cdots + a_n + a_{n+1} + a_{n+2} + \cdots$$

We can approximate the sum s with the partial sum

$$s_n = \sum_{i=1}^{n} a_i. = a_1 + a_2 + \cdots + a_n$$

The remainder (or error) is

$$R_n = s - s_n = \sum_{i=n+1}^{\infty} a_i = a_{n+1} + a_{n+2} + a_{n+3} + \cdots .$$

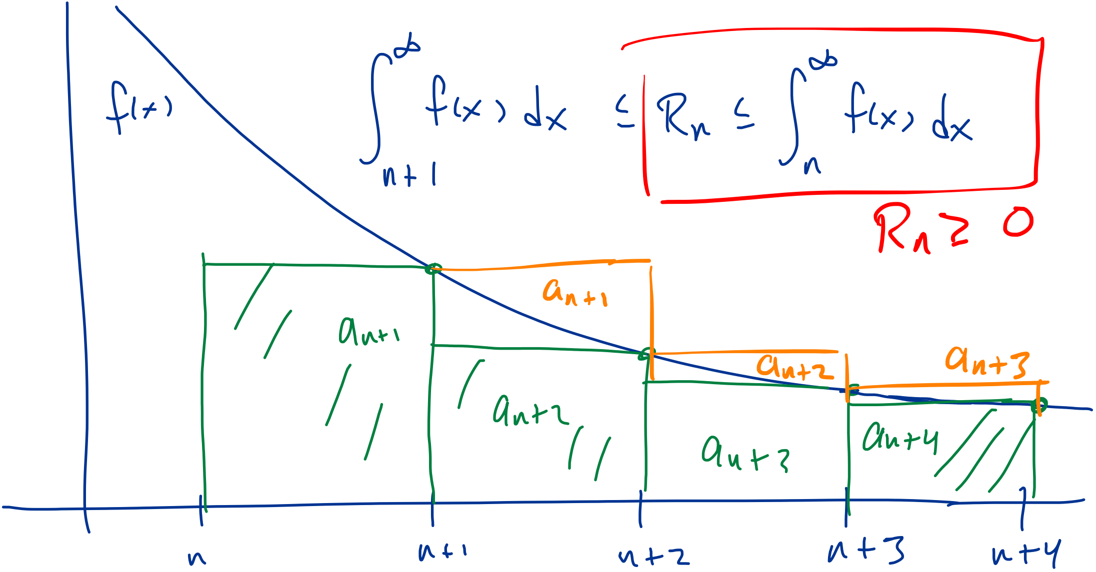
Question: How accurate is the approximation (how large is R_n)?

$$R_n = S - S_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

$$f(n) = a_n$$

$$\int_{n+1}^{\infty} f(x)\,dx \le \boxed{R_n \le \int_{n}^{\infty} f(x)\,dx}$$

$$R_n \ge 0$$



Estimates on the remainder

Suppose $a_k = f(k)$, where f is **continuous**, **positive**, and **decreasing** for $x \geq n$, and $s = \sum a_k$ is convergent. Then the remainder $R_n = s - s_n$ satisfies

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx.$$

How many terms, n, are required to approximate $s = \sum_{n=1}^{\infty} \frac{1}{n^2}$ by the partial sum

$$s_n = \sum_{k=1}^{n} \frac{1}{k^2}$$

want

with an error less than 0.001?

$$R_n = s - s_n \leq 0.001$$

By theorem

$$R_n \leq \int_n^\infty \frac{1}{x^2}\,dx$$

$$= \lim_{T\to\infty} \int_n^T \frac{1}{x^2}\,dx$$

$$= \lim_{T\to\infty} -\frac{1}{x}\Big]_n^T$$

$$= \lim_{T\to\infty} \left(\frac{1}{n} - \frac{1}{T}\right)$$

$$= \frac{1}{n} \le 0.001$$

want

$$\frac{1}{n} \le 0.001 = \frac{1}{1000}$$

$$\frac{1000}{n} \le 1 \quad \text{or} \quad \boxed{n \ge 1000}$$