Math 1272: Calculus II
11.5 Alternating series

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Alternating series

Recall the harmonic series:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

which **diverges**. How about the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \cdots ?$$

$\frac{1}{2}$

$\frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

$\frac{1}{5} - \frac{1}{6} = \frac{6}{30} - \frac{5}{30} = \frac{1}{30}$

$= \frac{1}{2} + \frac{1}{12} + \frac{1}{30} + \cdots$

General term $\frac{1}{n(n+1)}$ for n odd

$\approx \frac{1}{n^2}$

Looks like $\sum \frac{1}{n^2}$, which converges.

Alternating series test

Can be n

Consider an alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + b_7 - \cdots ,$$

where $b_n > 0$. If

- $0 < b_{n+1} \le b_n$ for all n (b_n is **positive** and **decreasing**), and
- $\lim_{n\to\infty} b_n = 0$,

then the series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ is convergent.

Proof $\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots$

Partial sums $S_N = \sum_{n=1}^{N} (-1)^{n-1} b_n$

$S_2 = b_1 - b_2 \geq 0$ since $b_2 \leq b_1$

$$S_4 = \underbrace{b_1 - b_2}_{\geq 0} + \underbrace{b_3 - b_4}_{\geq 0} \geq b_1 - b_2 = S_2$$

$$S_6 = \underbrace{b_1 - b_2}_{\geq 0} + \underbrace{b_3 - b_4}_{\geq 0} + \underbrace{b_5 - b_6}_{\geq 0} \geq b_1 - b_2 + b_3 - b_4 = S_4$$

The even partial sums S_{2N} are an increasing sequence

$$S_{2N+2} \geq S_{2N}$$

Bounded above?

$$S_6 = b_1 - b_2 + b_3 - b_4 + b_5 - b_6$$

$$= b_1 - \underbrace{(b_2 - b_3)}_{\geq 0} - \underbrace{(b_4 - b_5)}_{\geq 0} - b_6$$

$\leq b_1 - b_6 \leq b_1$ since $b_6 \geq 0$

In general

$$S_{2N} = b_1 - b_2 + b_3 - b_4 + \cdots + b_{2N-3} - b_{2N-2} + b_{2N-1} - b_{2N}$$

$$= b_1 - (b_2 - \cancel{b_3}) - (b_4 - \cancel{b_5}) - \cdots - (b_{2N-2} - \cancel{b_{2N-1}}) - b_{2N}$$

$$\leq b_1 - b_{2N} \leq b_1$$

$\{S_{2N}\}$ is increasing and bounded above by b_1

$\longrightarrow$ converges.

Write $S = \lim_{N \to \infty} S_{2N}$

Note $S_{2N+1} = S_{2N} + b_{2N+1}$

$$\lim_{N \to \infty} S_{2N+1} = \lim_{N \to \infty} S_{2N} + \lim_{N \to \infty} \cancelto{0}{b_{2N+1}}$$

$$= S$$

Follows that $\lim_{N \to \infty} S_N = S$, $\longrightarrow \sum (-1)^{n-1} b_n$ converges.

Example: Alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots$$

$b_n = \frac{1}{n}$ decreasing ✓ and $\lim_{n \to \infty} b_n = 0$ ✓

$\longrightarrow$ converges by AST

Example: Assess convergence/divergence of the alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^n 3n^2}{n^2-3}. \quad = \sum a_n$$

$$b_n = \frac{3n^2}{n^2-3} \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)} = \frac{3}{1-\frac{3}{n^2}} \xrightarrow[\infty]{n} 3$$

$a_n = (-1)^n b_n$ does not converge to zero

as $n \to \infty$

$\longrightarrow$ series diverges.

Example: Assess convergence/divergence of the alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^n 3n^2}{n^2 - 3}.$$

Example: Assess convergence/divergence of the alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}.$$

$b_n = \frac{\ln(n)}{n}$. To check decreasing, look at

$$\frac{d}{dx}\left(\frac{\ln(x)}{x}\right) = \frac{x \cdot \frac{1}{x} - \ln(x)}{x^2} = \frac{1-\ln(x)}{x^2}$$

Decreasing for $\ln(x) > 1$ or $x > e$

So $b_n = \frac{\ln(n)}{n}$ decreasing for $n \geq 3$.

and $\lim_{n\to\infty} \frac{\ln(n)}{n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$

l'Hospital

$\longrightarrow$ Converges by A.H. Series Test.

Estimation of alternating series

Consider an alternating series

$$s = \sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + b_7 - \cdots ,$$

where

- $0 < b_{n+1} \leq b_n$ for all n (b_n is **positive** and **decreasing**), and
- $\lim_{n\to\infty} b_n = 0$,

Define the partial sum

$$s_n = \sum_{i=1}^{n} (-1)^{i-1} b_i.$$

Then the remainder $R_n = s - s_n$ satisfies

$$|R_n| = |s - s_n| \leq b_{n+1}.$$

Find the sum of $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$ correct to 2 decimal places.

$$n! = n(n-1)(n-2)\cdots(3)\cdot(2)\cdot(1)$$

(want)

$$|R_n| \le b_{n+1} = \frac{1}{(n+1)!} \le \frac{1}{100} \quad \text{or} \quad (n+1)! \ge 100$$

$n=1$, $(n+1)! = 2! = 2$

$n=2$, $(n+1)! = 3! = 6$

$n=3$, " $4! = 24$

$n=4$, " $5! = 120$

Just need 4 terms

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \approx -1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24}$$ up to 2 decimal places.

$$= \frac{-24 + 12 - 4 + 1}{24}$$

$$= \frac{-15}{24} = \frac{-5}{8}$$