Math 1272: Calculus II
7.2 Trigonometric Integrals

Instructor: Jeff Calder
Office: 538 Vincent
Email: jcalder@umn.edu

http://www-users.math.umn.edu/~jwcalder/1272S19

Trigonometric integrals: warmup

Example 1. Use substitution find $\int \cos^3 x \sin x \, dx$.

$u = \cos x$, $\frac{du}{dx} = -\sin x$

$\rightarrow \boxed{du = -\sin x \, dx.}$

$$\int \cos^3 x \sin x \, dx = -\int u^3 du$$

$$= -\frac{1}{4} u^4 + C$$

$$= -\frac{1}{4} \cos^4 x + C$$

Trigonometric identities

All trigonometric identities can be derived from Euler's formula

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta.} \tag{1}$$

Here, $i = \sqrt{-1}$ so $i^2 = -1$ and θ is any real number. In particular

$$e^{i\pi} = \cos\pi + i\sin\pi = -1.$$

This is Euler's identity

$$\boxed{e^{i\pi} + 1 = 0.} \tag{2}$$

Proof of Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$.

$f(\theta) = \cos\theta + i\sin\theta$, $\quad \dfrac{df}{d\theta} = -\sin\theta + i\cos\theta$

$if(\theta) = i\cos\theta + i^2\sin\theta = i\cos\theta - \sin\theta = \dfrac{df}{d\theta}$

Hence $\quad f'(\theta) = if(\theta)$

$$\frac{d}{d\theta}\left(\frac{f(\theta)}{e^{i\theta}}\right) = \frac{e^{i\theta}f'(\theta) - f(\theta)\,ie^{i\theta}}{e^{2i\theta}}$$

$$= \frac{e^{i\theta}\,if(\theta) - f(\theta)\,ie^{i\theta}}{e^{2i\theta}} = 0$$

Hence $\dfrac{f(\theta)}{e^{i\theta}} = C$ or $f(\theta) = Ce^{i\theta}$

$\therefore \quad \cos\theta + i\sin\theta = Ce^{i\theta}$

Take $\theta = 0$, $\cos(0) = 1$, $\sin(0) = 0$

and $e^{i\cdot 0} = 1$, so

$1 + \cancel{i0} = C\cdot\cancel{1} \Rightarrow C = 1$ $\blacksquare$

Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$

Use $\boxed{e^{ix} e^{iy} = e^{ix+iy} = e^{i(x+y)}}$

$$e^{ix} e^{iy} = \cos(x+y) + i\sin(x+y) \qquad 1$$

$$e^{ix} e^{iy} = (\cos x + i\sin x)(\cos y + i\sin y)$$

$$= \cos x\cos y + i\cos x\sin y$$

$$+ i\sin x\cos y + \underbrace{i\sin x \cdot i\sin y}_{= i^2 \sin x\sin y = -\sin x\sin y}$$

$$e^{ix} e^{iy} = \cos x \cos y - \sin x \sin y$$

$$+ i(\sin x \cos y + \cos x \sin y)$$

$$e^{ix} e^{iy} = \cos(x+y) + i \sin(x+y)$$

Fact: if $a + bi = c + di \Rightarrow \begin{array}{l} a = c \\ b = d \end{array}$

Short proof: $a - c = i(d - b)$

If $b \neq d$ then $i = \dfrac{a-c}{d-b}$ contradiction

So $b = d$ and $a = c$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$1 = e^{i\theta} e^{-i\theta} = (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)$$

$$= \cos^2\theta \cancel{- i\cos\theta\sin\theta}$$

$$\cancel{+ i\sin\theta\cos\theta} + i\sin\theta \cdot (-i\sin\theta)$$

$$= \cos^2\theta + \sin^2\theta .$$

Trigonometric identities

Angle sum formulas

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

Replace y with $-y$ and use $\sin(-y) = -\sin(y)$

$\cos(-y) = \cos(y)$

$$\boxed{\begin{aligned}\sin(x-y) &= \sin x \cos y - \cos x \sin y \\ \cos(x-y) &= \cos x \cos y + \sin x \sin y\end{aligned}}$$

Trigonometric identities

Angle sum formulas

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y \quad (1)$$

Angle difference formulas

$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$
$$\cos(x-y) = \cos x \cos y + \sin x \sin y \quad (2)$$

(1) + (2) : $\cos(x+y) + \cos(x-y) = 2\cos x \cos y$

$$\cos x \cos y = \frac{1}{2}\left[\cos(x+y) + \cos(x-y)\right]$$

Trigonometric identities

Angle sum formulas

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

Angle difference formulas

$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$
$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$

Product formulas

$$\sin x \cos y = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]$$
$$\cos x \cos y = \frac{1}{2}\left[\cos(x+y) + \cos(x-y)\right]$$
$$\sin x \sin y = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$$

Trigonometric identities

$$\sin(nt)\sin(mt) \quad \underbrace{\;nt\;}_{x} \quad \underbrace{\;mt\;}_{y}$$

$$= \frac{1}{2}\left[\cos(nt-mt) - \cos(nt+mt)\right]$$

Angle sum formulas

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

Angle difference formulas

$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$

$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$

Product formulas

$$\sin x \cos y = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]$$

$$\cos x \cos y = \frac{1}{2}\left[\cos(x+y) + \cos(x-y)\right] \; (*)$$

$$\sin x \sin y = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$$

Don't forget

$$\cos^2 x = \frac{1}{2}\left(\cos(2x) + 1\right)$$

$$\sin^2 x + \cos^2 x = 1$$

Example 2. Evaluate $\int_0^\pi \cos^2 x\, dx$.

$\cos^2 x = \frac{1}{2}(\cos(2x) + 1)$

$\int_0^{\pi} \frac{1}{2}(\cos(2x)+1)\,dx$,

$\boxed{u = 2x}$

$du = 2\,dx$, $dx = \frac{1}{2}du$

$= \int_0^{2\pi} \frac{1}{2}(\cos(u)+1)\cdot\frac{1}{2}du$

$x: 0 \longrightarrow \pi$

$u = 2x$

$u: 0 \longrightarrow 2\pi$

$= \frac{1}{4}\int_0^{2\pi} \cos(u) + 1\, du$

$= \frac{1}{4}\left[\sin(u) + u\right]\Big|_0^{2\pi} = \frac{1}{4}(2\pi - 0) = \frac{\pi}{2}.$

Example 3. Find $\int \cos^3 x \sin^2 x \, dx$.

Put in form $\int \sin^m x \cos x \, dx$

or $\int \cos^m x \sin x \, dx$

Use $\cos^2 x + \sin^2 x = 1$

$\cos^2 x = 1 - \sin^2 x$

$$\int \cos^3 x \sin^2 x \, dx = \int \cos x \,(1 - \sin^2 x) \sin^2 x \, dx$$

$u = \sin x$

$\frac{du}{dx} = \cos x$

$du = \cos x \, dx$

$$= \int (1 - u^2) u^2 \, du$$

$$= \int u^2 - u^4 \, du$$

$$= \frac{1}{3} u^3 - \frac{1}{5} u^5 + C$$

$u = \sin x$ | $= \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$

$$\int \cos^3 x \sin^2 x \, dx = \int \cos^3 x \left(1 - \cos^2 x\right) dx$$

$\sin^2 x = 1 - \cos^2 x$ | $u = \cos x$, $du = -\sin x \, dx$

Example 4. Evaluate $\int_{-\pi}^{\pi} \sin(nt)\sin(mt)\,dt$ for positive integers n, m.

$$\boxed{\begin{aligned}\sin(nt)\sin(mt) &= \frac{1}{2}\left[\cos(nt-mt) - \cos(nt+mt)\right] \\ &= \frac{1}{2}\left[\cos((n-m)t) - \cos((n+m)t)\right]\end{aligned}}$$

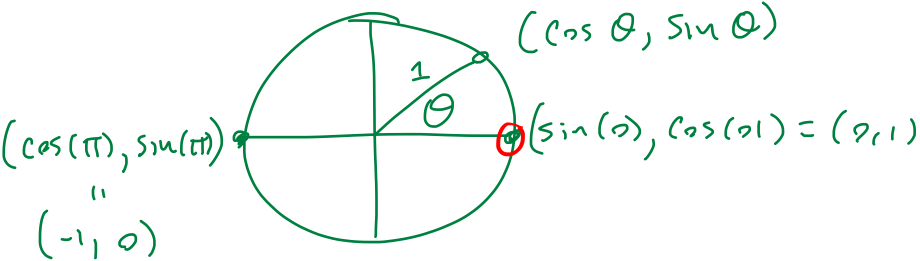
Case 1: $k=0$: $\displaystyle\int_{-\pi}^{\pi} \cos(kt)\,dt = \int_{-\pi}^{\pi} 1\,dt = 2\pi$

$u = kt$

Case 2: $k \neq 0$ integer

$$\int_{-\pi}^{\pi} \cos(kt)\,dt = \frac{\sin(kt)}{k}\Bigg]_{-\pi}^{\pi} = \frac{\sin(k\pi) - \sin(-k\pi)}{k} = 0$$

Claim $\sin(k\pi) = 0$ for integers k.



$$\sin(0) = 0 = \sin(\pi)$$

Sin is 2π-periodic, $\sin(2\pi) = \sin(0) = 0$

$$\sin(3\pi) = \sin(\pi) = 0$$

$$\vdots$$

$$\sin(k\pi) = 0$$

$$\sin(nt)\sin(mt) = \frac{1}{2}\left[\cos(nt-mt) - \cos(nt+mt)\right]$$

$$= \frac{1}{2}\left[\cos((n-m)t) - \cos((n+m)t)\right]$$

$$\int_{-\pi}^{\pi} \sin(nt)\sin(mt)\,dt = \frac{1}{2}\int_{-\pi}^{\pi} \cos((n-m)t)\,dt$$

$$\underbrace{- \frac{1}{2}\int_{-\pi}^{\pi} \cos((n+m)t)\,dt}_{=0}$$

$$\boxed{\begin{array}{l} n+m \neq 0 \\ k = n-m \end{array}}$$

$$= \begin{cases} 2\pi, & n = m \quad (k=0) \\ 0, & n \neq m \end{cases}$$

<u>**Fourier Series**</u>

$$\sec\theta = \frac{1}{\cos\theta}$$

Example 5. Find $\int \tan^3\theta \sec^5\theta \, d\theta$.

Two substitutions

$$\boxed{\begin{array}{l} (1)\ u = \tan\theta, \quad du = \sec^2\theta \, d\theta \\ (2)\ u = \sec\theta, \quad du = \tan\theta\sec\theta \, d\theta \end{array}}$$

(2) $$\frac{du}{d\theta} = \frac{d}{d\theta}\frac{1}{\cos\theta} = \frac{d}{d\theta}(\cos\theta)^{-1} = -1(\cos\theta)^{-2}(-\sin\theta) = \frac{\sin\theta}{\cos^2\theta}$$

(1): $$\frac{du}{d\theta} = \frac{d}{d\theta}\frac{\sin\theta}{\cos\theta} = \frac{\cos\theta\cos\theta - \sin\theta(-\sin\theta)}{\cos^2\theta}$$

$$= \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} = \sec^2\theta$$

$$1 + \tan^2\theta = 1 + \frac{\sin^2\theta}{\cos^2\theta} = \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}$$

$$= \frac{1}{\cos^2\theta} = \sec^2\theta$$

$$\boxed{1 + \tan^2\theta = \sec^2\theta}$$

Try ② $u = \sec\theta$, $du = \sec\theta \tan\theta$

$$\int \tan^3\theta \sec^5\theta \, d\theta = \int \tan^2\theta \, u^4 \, du \qquad \tan^2\theta = u^2 - 1$$

$$= \int (u^2 - 1) u^4 \, du \; \ldots\ldots$$

Example: $\int \tan^6\theta \sec^4\theta \, d\theta$, $u = \tan\theta$

$du = \sec^2\theta \, d\theta$

$= \int \tan^6\theta \sec^2\theta \, du$, $\sec^2\theta = 1 + u^2$

$\tan^6\theta = u^6$

$= \int u^6 (1+u^2) \, du$

$= \ldots$