Math 1272: Calculus II
11.10 Taylor Series

Instructor: Jeff Calder
Office: 538 Vincent
Email: jcalder@umn.edu

http://www-users.math.umn.edu/~jwcalder/1272S19

Representing functions as power series

Given an arbitrary function $f(x)$, can we find a power series representation

$$(*) \qquad f(x) = \sum_{n=0}^{\infty} c_n x^n ? \qquad c_n = ?$$

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$$

If $(*)$ holds then $f(0) = c_0$ ✓ $\quad 0! = 1$

$$f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 + \cdots$$

$$f'(0) = c_1 = \frac{f'(0)}{1}$$

$$f''(x) = 2c_2 + 3\cdot 2c_3x + 4\cdot 3c_4x^2 + 5\cdot 4c_5x^3 + \cdots$$

$$f''(0) = 2c_2 \rightsquigarrow c_2 = \frac{f''(0)}{2\cdot 1}$$

$$f'''(x) = 3\cdot 2c_3 + 4\cdot 3\cdot 2c_4x + 5\cdot 4\cdot 3c_5x^2 + \cdots$$

$$f'''(0) = 3\cdot 2c_3 \rightsquigarrow c_3 = \frac{f'''(0)}{3\cdot 2\cdot 1}$$

$$f^{(4)}(x) = 4\cdot 3\cdot 2c_4 + 5\cdot 4\cdot 3\cdot 2c_5x + \cdots$$

$$f^{(4)}(0) = 4\cdot 3\cdot 2c_4 \rightsquigarrow c_4 = \frac{f^{(4)}(0)}{4\cdot 3\cdot 2\cdot 1}$$

$$f^{(5)}(0) = 5\cdot 4\cdot 3\cdot 2\, c_5 \Rightarrow c_5 = \frac{f^{(5)}(0)}{5\cdot 4\cdot 3\cdot 2\cdot 1}$$

General formula

$$c_n = \frac{f^{(n)}(0)}{n!}$$

$$n! = 1\cdot 2\cdot 3\cdot 4\cdot \cdots (n-1)\cdot n$$

Theorem If f has a power series representation at a

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \quad |x-a| < R$$

then

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

The representation

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n, \quad |x-a| < R$$

is called the **Taylor series** of f centered at a.

When $a = 0$ it is often called the **Maclaurin series**.

$a = 0$

Example Find the Maclaurin series for $f(x) = e^x$, and the radius of convergence.

for all $n \geq 0$, $f^{(n)}(x) = e^x$

$f^{(n)}(0) = e^0 = 1$

Maclaurin Series $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

Maclaurin Series for e^x is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

Ratio test: $a_n = \frac{x^n}{n!}$, $\frac{1}{|a_n|} = \frac{n!}{|x|^n}$

$|a_{n+1}| = \frac{|x|^{n+1}}{(n+1)!}$

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} \frac{n!}{|x|^n} = \lim_{n\to\infty} \frac{n!}{(n+1)!} |x|$$

$$= \lim_{n\to\infty} \frac{|x|}{n+1} = 0$$

$$\frac{n!}{(n+1)!} = \frac{\cancel{1\cdot 2\cdot 3\cdot 4 \cdots n}}{\cancel{1\cdot 2\cdot 3\cdot 4\cdots n}\,(n+1)}$$

Converges for all x

Example

Maclaurin Series $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

$$f(x) = e^{-\frac{1}{x^2}} \qquad (f(0) = 0)$$

Fact: $f^{(n)}(0) = 0$ for all $n \geq 0$

Maclaurin Series for $f(x)$ is $\sum_{n=0}^{\infty} \frac{0}{n!} x^n = 0$

Maclaurin Series does not converge to $f(x)$ for $x \neq 0$.

Convergence of Taylor series to $f(x)$

To examine convergence of Taylor series to $f(x)$, we consider the partial sums

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i,$$

$n = 0$

$T_0(x) = f(a)$

and remainder

$$R_n(x) = f(x) - T_n(x).$$

$n = 1$

$f(x) = T_n(x) + R_n(x)$

$T_1(x) = f(a) + f'(a)(x-a)$

The Taylor series converges to $f(x)$ if

$$\lim_{n\to\infty} T_n(x) = f(x) \quad \text{or} \quad \lim_{n\to\infty} R_n(x) = 0.$$

Question: How can we show that $R_n \to 0$?

Fundamental Theorem

$$\int_a^x f'(t)\,dt = f(x) - f(a)$$

$$f(x) = f(a) + \int_a^x f'(t)\,dt$$

$$= T_0(x) + \underbrace{\int_a^x f'(t)\,dt}_{R_0(x)}$$

Assume $|f'(t)| \leq M$ for $|t-a| \leq |x-a|$

Then $|R_0(x)| = \left| \int_a^x f'(t)\,dt \right|$, $x > a$

$$\le \int_a^x \underbrace{|f'(t)|}_{\le M} dt$$

$$\le M \int_a^x dt = M(x-a)$$

$$= M|x-a|$$

$$|R_0(x)| \le M|x-a| \quad , \quad |f'(t)| \le M$$

for $|t-a| \le |x-a|$

We have

$$f(x) = f(a) + \int_a^x f'(t)\,dt$$

(Fund. Theorem of Calc)

$$\int_a^t f''(s)\,ds = f'(t) - f'(a)$$

$$f'(t) = f'(a) + \int_a^t f''(s)\,ds$$

$$f(x) = f(a) + \int_a^x \left[f'(a) + \int_a^t f''(s)\,ds \right] dt$$

$$= f(a) + \int_a^x f'(a)\,dt + \int_a^x \int_a^t f''(s)\,ds\,dt$$

$$= f(a) + f'(a)\int_a^x dt + \int_a^x \int_a^t f''(s)\,ds\,dt$$

$$= \underbrace{f(a) + f'(a)(x-a)}_{T_1(x)} + \underbrace{\int_a^x \int_a^t f''(s)\,ds\,dt}_{R_1(x)}$$

tangent line = $T_1(x)$

Assume $|f''(s)| \le M$ for $|s-a| \le |x-a|$

Then

$$|R_1(x)| \leq M \int_a^x \int_a^t 1 \, ds \, dt, \quad \begin{matrix} x > a \\ t > a \end{matrix}$$

$$= M \int_a^x (t-a) \, dt$$

$$= M \left. \frac{1}{2}(t-a)^2 \right]_a^x$$

$$= \frac{M}{2}(x-a)^2$$

Taylor's inequality

If $|f^{(n+1)}| \leq M$ for $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for } |x-a| < d.$$

Proof by induction.

$M = M(n)$

$f(x) = e^x, \quad f^{(n)}(x) = e^x, \quad f^{(n)}(0) = 1$

Exercise: Prove that e^x is equal to its Maclaurin series.

Maclaurin series $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

If $|x| \le d$, $-d \le x \le d$, then

$$|f^{(n+1)}(x)| = e^x \le e^d = M$$

$$|R_n(x)| \le \frac{M}{(n+1)!} |x|^{n+1} = \frac{e^d \, |x|^{n+1}}{(n+1)!}$$

$a = 0$

Consider $\sum_{n=0}^{\infty} \frac{|x|^n}{n!}$. Ratio test

$$\lim_{n\to\infty} \underbrace{\frac{|x|^{n+1}}{(n+1)!} \frac{n!}{|x|^n}}_{\left|\frac{a_{n+1}}{a_n}\right|} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1$$

$\rightarrow$ Series converges by Ratio test

$\longrightarrow$ $\lim_{n\to\infty} \frac{|x|^n}{n!} = 0$ $\longrightarrow$ $\lim_{n\to\infty} R_n(x) = 0$

Hence $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x

Recall, $f(x) = e^{-\frac{1}{x^2}} = \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{x^2}\right)^n}{n!}$ ← fun

$e^{-\frac{1}{x^2}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\, x^{2n}}$ $x \neq 0$

↑ Laurent series

"Taylor series centered at $x = \infty$"

Exercise: Find the Taylor series for $f(x) = e^x$ at $a = 2$.

$$c_n = \frac{f^{(n)}(a)}{n!} = \frac{e^a}{n!}$$

$a = 2$

Taylor series is $\displaystyle\sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=0}^{\infty} \frac{e^a (x-a)^n}{n!}$

$$e^x = e^a e^{x-a} = e^a \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!}$$

already proved conversion

Exercise: Find the Maclaurin series for $\sin x$ and prove that it represents $\sin x$ for all x.

$$c_n = \frac{f^{(n)}(0)}{n!}, \quad f(x) = \sin(x).$$

$$f^{(0)}(0) = f(0) = \sin(0) = \boxed{0}, \quad f(x) = \sin(x)$$

$$f^{(1)}(0) = \cos(0) = \boxed{1}, \quad f'(x) = \cos(x)$$

$$f^{(2)}(0) = -\sin(0) = \boxed{0} \quad f''(x) = -\sin(x)$$

$$f^{(3)}(0) = -\cos(0) = \boxed{-1} \quad f^{(3)}(x) = -\cos(x)$$

$$f^{(4)}(0) = \sin(0) = \boxed{0} \quad f^{(4)}(x) = \sin(x)$$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{1}{2!} f''(0) x^2 + \frac{1}{3!} f'''(0) x^3 + \frac{1}{4!} f^{(4)}(0) x^4 + \frac{1}{5!} f^{(5)}(0) x^5 + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$n=0$, $n=1$, $n=2$

Maclaurin Series

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Does this converge to $\sin(x)$?

Answer: Yes, since $|f^{(n)}(x)| \le 1 = M$

So $$|R_n(x)| \le \frac{M|x|^{n+1}}{(n+1)!} = \frac{|x|^{n+1}}{(n+1)!}$$

$$\longrightarrow \lim_{n\to\infty} R_n(x) = 0$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Exercise: Find the Maclaurin series for $\cos x$.

Since $\frac{d}{dx} \sin x = \cos x$

$$\cos x = \frac{d}{dx} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1) x^{2n}}{(2n+1)!} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}}$$

$$(2n+1)! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (2n)(2n+1) = (2n)!\,(2n+1)$$

Ex: Prove $e^{ix} = \cos x + i\sin x$ (Euler's Formula)

using Maclaurin series for

e^{ix}, $\cos x$, $\sin x$

Exercise: Find the Maclaurin series for $f(x) = (1+x)^k$, where k is any real number.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$f(0) = 1$

$f'(x) = k(1+x)^{k-1} \rightsquigarrow f'(0) = k$

$f''(x) = k(k-1)(1+x)^{k-2} \rightsquigarrow f''(0) = k(k-1)$.

$f'''(x) = k(k-1)(k-2)(1+x)^{k-3} \rightsquigarrow f'''(0) = k(k-1)(k-2)$

$f^{(4)}(x) = k(k-1)(k-2)(k-3)(1+x)^{k-4}$ $n=4$

$\rightsquigarrow f^{(4)}(0) = k(k-1)(k-2)(k-3)$

In general $f^{(n)}(0) = k(k-1)(k-2)\cdots(k-(n-1))$

Maclaurin series for $(1+x)^k$ is

$$\sum_{n=0}^{\infty} \frac{k(k-1)(k-2)\cdots(k-(n-1))}{n!} x^n$$

$$\binom{k}{n} := \frac{k(k-1)\cdots(k-(n-1))}{n!}$$

$$= \sum_{n=0}^{\infty} \binom{k}{n} x^n \quad \text{(Binomial series).}$$

Exercise: Find the Maclaurin series for $f(x) = \frac{1}{\sqrt{4-x}}$ and its radius of convergence.

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n \quad \text{(Binomial series)}$$

$$\frac{1}{\sqrt{4-x}} = \frac{1}{\sqrt{4}} \cdot \frac{1}{\sqrt{1-\frac{x}{4}}} = \frac{1}{2}\left(1 + \left(-\frac{x}{4}\right)\right)^{-\frac{1}{2}}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \left(-\frac{x}{4}\right)^n$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \binom{-\frac{1}{2}}{n} \frac{x^n}{4^n}$$

$$\binom{-\frac{1}{2}}{n} = -\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\underbrace{\left(-\frac{1}{2}-(n-1)\right)}$$

$$\frac{-1-2n+2}{2} = -\left(\frac{2n-1}{2}\right)$$

$$= \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\left(\frac{2n-1}{2}\right)\right)$$

$$= (-1)^n \frac{3\cdot 5\cdot 7\cdot\cdots\cdot(2n-1)}{2^n}$$

Exercise: Evaluate $\int_0^1 e^{-x^2}\,dx$ as an infinite series. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\int_0^1 e^{-x^2}\,dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 x^{2n}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+1}}{2n+1}\Bigg]_0^1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{1}{2n+1}$$

Multiplication and division of power series

Find a power series representation for $e^x \sin x$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{6} + \frac{x^5}{5!} + \cdots$$

$$e^x \sin x = 1 \cdot x - 1 \cdot \frac{x^3}{6} + \cdots + x^2 - \cancel{\frac{x^4}{6}} + \cdots$$

$$+ \frac{x^3}{2} + \cdots$$

$$= x + x^2 - \frac{x^3}{6} + \frac{x^3}{2} + \cdots$$

$$= x + x^2 + \frac{1}{3}x^3 + \cdots$$

$$\left(\sum a_n x^n\right)\left(\sum b_n x^n\right) \neq \sum a_n b_n x^{2n}$$

$$(a_1 + a_2 x)(b_1 + b_2 x) = a_1 b_1 + a_2 b_2 x^2$$

$$+ a_1 b_2 x + a_2 b_1 x$$

$$\triangleright = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n$$