Math 1272: Calculus II
12.2 Vectors

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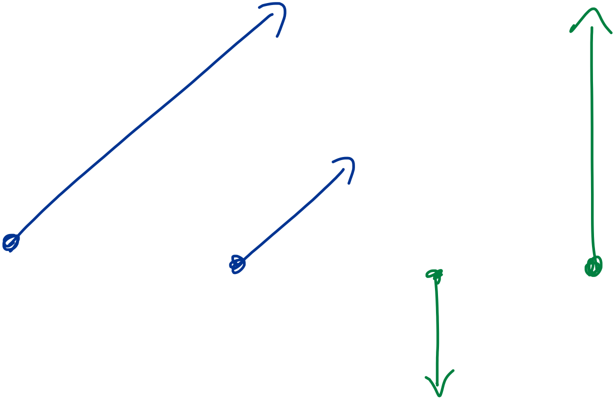
Vectors

Definition: A vector is a quantity (such as displacement or force) that has both **magnitude** and **direction**.



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Definition: A vector is a quantity (such as displacement or force) that has both **magnitude** and **direction**.

A vector $\mathbf{a}$ in $\mathbb{R}^3$ is represented by coordinates as

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle.$$

- The length (magniutude) of $\mathbf{a}$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

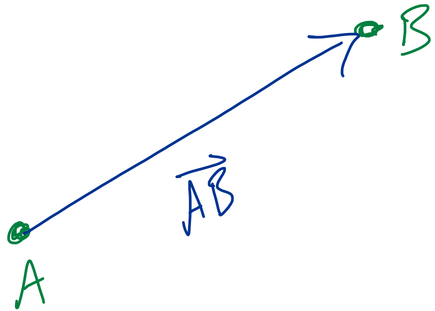
- The direction of $\mathbf{a}$ is the unit vector

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}.$$

Definition: Given points $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$, the vector $\mathbf{a}$ from A to B is

$$\mathbf{a} = \overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle.$$

- The **direction** of $\mathbf{a}$ points from A to B.
- The **magnitude** of $\mathbf{a}$ is the distance from A to B.



Operations with vectors

b_1

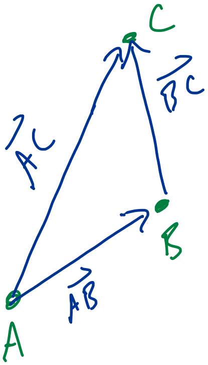
Addition and subtraction: If $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ then

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$
$$\mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle.$$

Scalar multiplication: If $\mathbf{a} = (a_1, a_2, a_3)$ and $\lambda \in \mathbb{R}$ then

$$\lambda \mathbf{a} = \langle \lambda a_1, \lambda a_2, \lambda a_3 \rangle.$$

Exercise: Show that $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.



$A = (a_1, a_2, a_3)$, $B = (b_1, b_2, b_3)$

$C = (c_1, c_2, c_3)$

$\overrightarrow{AB} = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle$

$\overrightarrow{BC} = \langle c_1 - b_1, c_2 - b_2, c_3 - b_3 \rangle$

$\overrightarrow{AB} + \overrightarrow{BC} = \langle c_1 - a_1, c_2 - a_2, c_3 - a_3 \rangle$

$= \overrightarrow{AC}$.

Illustration of adding vectors

$\vec{c} = \vec{a} + \vec{b}$

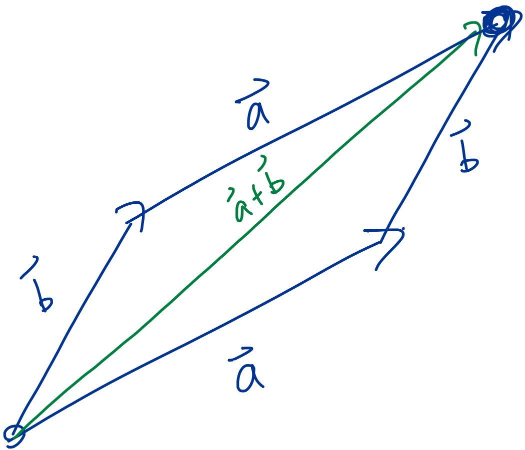
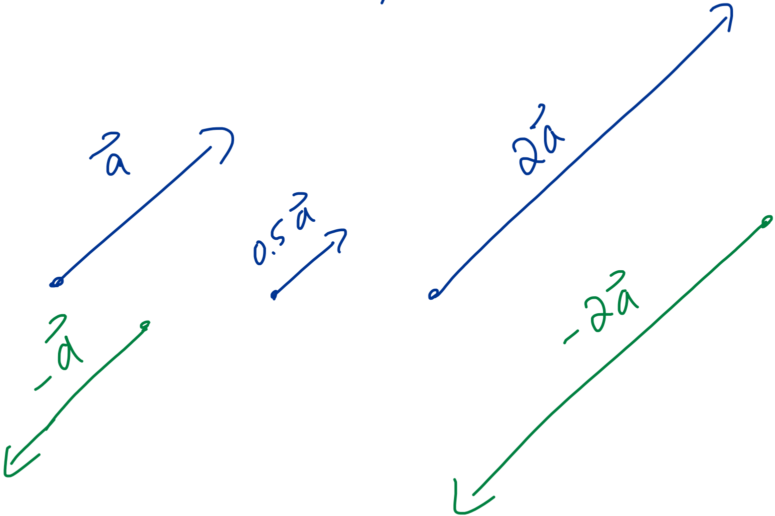


Illustration of scalar multiplication

$\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\lambda \vec{a} = \langle \lambda a_1, \lambda a_2, \lambda a_3 \rangle$



Vector spaces and properties of vectors

The set of all n-dimensional vectors of the form

$$\mathbf{a} = \langle a_1, a_2, \ldots, a_n \rangle$$

is denoted V_n.

If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are vectors in V_n and c, d are scalars then

$$\begin{aligned} \mathbf{a} + \mathbf{b} &= \mathbf{b} + \mathbf{a} & \mathbf{a} + (\mathbf{b} + \mathbf{c}) &= (\mathbf{a} + \mathbf{b}) + \mathbf{c} \\ \mathbf{a} + \mathbf{0} &= \mathbf{a} & \mathbf{a} + (-\mathbf{a}) &= \mathbf{0} \\ c(\mathbf{a} + \mathbf{b}) &= c\mathbf{a} + c\mathbf{b} & (c + d)\mathbf{a} &= c\mathbf{a} + d\mathbf{a} \\ (cd)\mathbf{a} &= c(d\mathbf{a}) & 1\mathbf{a} &= \mathbf{a}. \end{aligned}$$

Here $\mathbf{0} = \langle 0, 0, \ldots, 0 \rangle$.

Standard basis vectors

The standard basis vectors in $\mathbb{R}^3$ are

$$\mathbf{i} = \langle 1, 0, 0 \rangle, \ \mathbf{j} = \langle 0, 1, 0 \rangle, \ \text{ and } \mathbf{k} = \langle 0, 0, 1 \rangle.$$

Every vector in $\mathbb{R}^3$ can be expressed in terms of $\mathbf{i}, \mathbf{j}, \mathbf{k}$:

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}.$$

$$a_1 \langle 1, 0, 0 \rangle + a_2 \langle 0, 1, 0 \rangle$$
$$+ a_3 \langle 0, 0, 1 \rangle$$

Exercise: If $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} + 7\mathbf{k}$, express $2\mathbf{a} + 3\mathbf{b}$ in terms of $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

$$2\vec{a} + 3\vec{b} = 2(\hat{i} + 2\hat{j} - 3\hat{k}) + 3(4\hat{i} + 7\hat{k})$$

$$= 14\hat{i} + 4\hat{j} + 15\hat{k}$$

$$= \langle 14, 4, 15 \rangle$$

Unit vectors

A **unit vector** is a vector whose length is 1. The standard basis vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors.

$$\hat{i} = \langle 1, 0, 0 \rangle, \quad |\hat{i}| = \sqrt{1^2 + 0^2 + 0^2} = 1$$

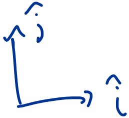
$$|\hat{j}| = 1, \quad |\hat{k}| = 1$$

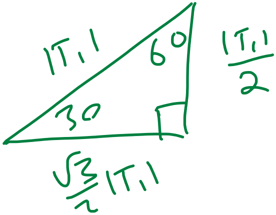
Exercise: Find a unit vector in the direction of the vector $\underbrace{2\mathbf{i} - \mathbf{j} - 2\mathbf{k}}_{\vec{a}}$.

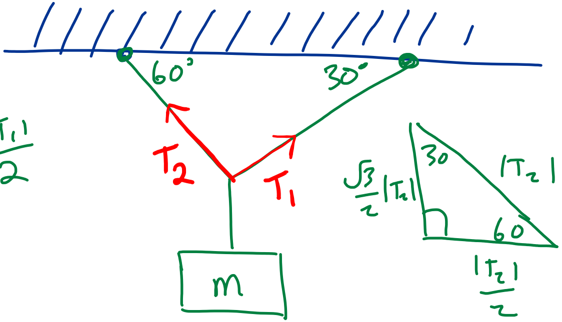
$$|\vec{a}| = \sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3$$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{3}(2\hat{i} - \hat{j} - 2\hat{k}) = \frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}$$

Exercise: Tension in wires.







$$T_1 = \frac{\sqrt{3}}{2}|T_1|\,\hat{i} + \frac{|T_1|}{2}\,\hat{j}$$

$$F_g = -mg\,\hat{j}$$

$$T_1 = \langle a, b\rangle$$

$$T_2 = -\frac{|T_2|}{2}\,\hat{i} + \frac{\sqrt{3}}{2}|T_2|\,\hat{j}$$

$$|T_1| = \sqrt{a^2+b^2}$$

$$T_1 + T_2 + F_g = 0$$

$$T_2 = \langle c, d\rangle$$

$$\underbrace{\left(\frac{\sqrt{3}}{2}|T_1| - \frac{|T_2|}{2}\right)}_{=0}\hat{i} + \underbrace{\left(\frac{|T_1|}{2} + \frac{\sqrt{3}}{2}|T_2| - mg\right)}_{=0}\hat{j} = 0$$

$$\sqrt{3}\left(\sqrt{3}\,|T_1| - |T_2|\right) = 0$$

$$+ \quad |T_1| + \sqrt{3}\,|T_2| = 2mg$$

$$4|T_1| = 2mg \longrightarrow \boxed{|T_1| = \frac{mg}{2}}$$

$$\boxed{|T_2| = \sqrt{3}\,|T_1| = \frac{\sqrt{3}}{2}mg}$$