Math 1272: Calculus II
12.3 The Dot Product

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The dot product

Definition: The **dot product** between vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ is defined as

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Exercise: Compute the dot product between $\mathbf{a} = \langle 1, 2, 3 \rangle$ and $\mathbf{b} = \langle -1, 0, 1 \rangle$.

$$\vec{a} \cdot \vec{b} = 1 \cdot (-1) + 2 \cdot (0) + 3 \cdot 1$$

$$= -1 + 3 = 2$$

Properties of dot products

If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are vectors and λ a scalar, then

$$\mathbf{a} \cdot \mathbf{a} = |a|^2 \qquad \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} \qquad (\lambda \mathbf{a}) \cdot \mathbf{b} = \lambda(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (\lambda \mathbf{b})$$

$$\mathbf{0} \cdot \mathbf{a} = 0$$

$$\vec{a} = \langle a_1, a_2, a_3 \rangle$$

$$\vec{a} \cdot \vec{a} = a_1 a_1 + a_2 a_2 + a_3 a_3$$

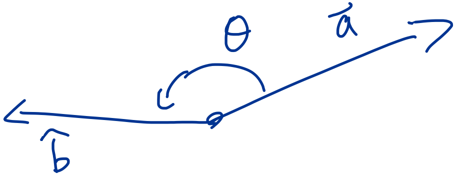
$$= a_1^2 + a_2^2 + a_3^2 = |\vec{a}|^2$$

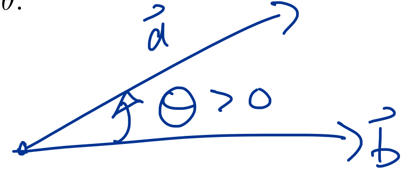
$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Angles between vectors

Theorem: If θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$ then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta.$$





Angles between vectors

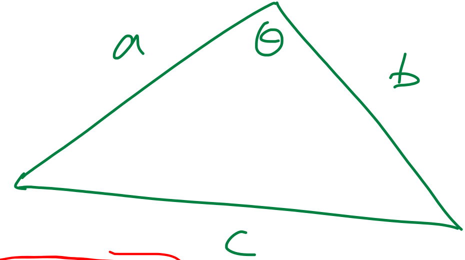
Theorem: If θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$ then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta.$$

Corollary: If θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$ then

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}.$$

Law of cosines



$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

$$|\vec{a}-\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$$

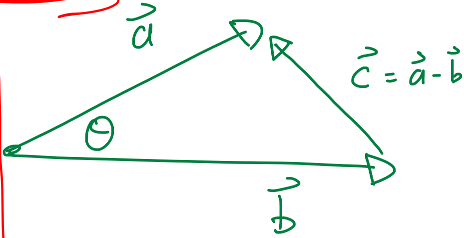
$$|\vec{a}-\vec{b}|^2 = (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})$$

$$= \vec{a}\cdot(\vec{a}-\vec{b}) - \vec{b}\cdot(\vec{a}-\vec{b})$$

$$= \vec{a}\cdot\vec{a} - \vec{a}\cdot\vec{b} - \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b}$$

$$= |\vec{a}|^2 + |\vec{b}|^2 - 2\,\vec{a}\cdot\vec{b}$$

$$\rightarrow \vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$



$$\vec{b} + \vec{c} = \vec{a}$$

$$\vec{c} = \vec{a} - \vec{b}$$

Exercise: Find the angle between $\mathbf{a} = \langle 2, 2, -1 \rangle$ and $\mathbf{b} = \langle 5, -3, 2 \rangle$.

$$\vec{a} \cdot \vec{b} = 2(5) + 2(-3) + (-1)(2)$$

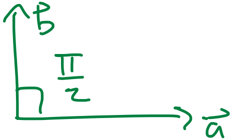
$$= 10 - 6 - 2 = 2$$

$$|\vec{a}| = \sqrt{2^2 + 2^2 + (-1)^2} = \sqrt{9} = 3$$

$$|\vec{b}| = \sqrt{5^2 + (-3)^2 + 2^2} = \sqrt{38}$$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|} = \frac{2}{3\sqrt{38}}$$

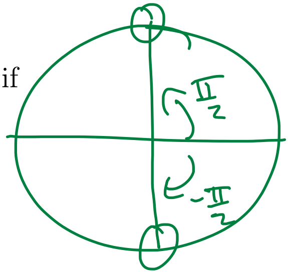
Orthogonality



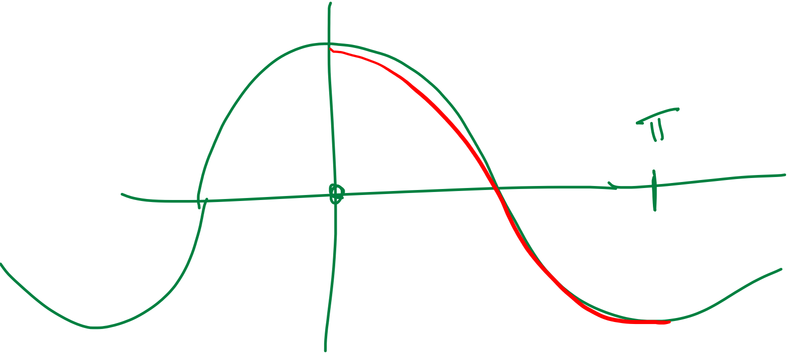
Two vectors **a** and **b** are orthogonal (perpendicular) if the angle between them is $\theta = \pi/2$. Recall $\cos \pi/2 = 0$ and

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta.$$

Hence, two vectors **a** and **b** are **orthogonal** if and only if



$$\mathbf{a} \cdot \mathbf{b} = 0.$$



Exercise: Show that $\underbrace{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}_{\vec{a}}$ is perpendicular to $\underbrace{5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}}_{\vec{b}}$.

$$\vec{a} \cdot \vec{b} = 2(5) + 2(-4) + (-1)(2)$$

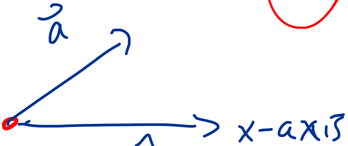
$$= 10 - 8 - 2 = 0$$

$\Longrightarrow$ $\vec{a}$ and $\vec{b}$ are perpendicular

Direction angles

The **direction angles** of a nonzero vector **a** are the angles α, β, and γ (in $[0, \pi]$) that **a** makes with the positive x-,y-, and z-axes, respectively.

$$\cos\alpha = \frac{a_1}{|\mathbf{a}|}, \quad \cos\beta = \frac{a_2}{|\mathbf{a}|}, \quad \text{and} \quad \cos\gamma = \frac{a_3}{|\mathbf{a}|}.$$



$\hat{i} = \langle 1, 0, 0 \rangle$

$|\hat{i}| = 1$

$$\cos\alpha = \frac{\vec{a} \cdot \hat{i}}{|\vec{a}||\hat{i}|} = \frac{a_1}{|\vec{a}|}$$

Direction angles

The **direction angles** of a nonzero vector $\mathbf{a}$ are the angles α, β, and γ (in $[0, \pi]$) that $\mathbf{a}$ makes with the positive x-,y-, and z-axes, respectively.

$$\cos\alpha = \frac{a_1}{|\mathbf{a}|}, \quad \cos\beta = \frac{a_2}{|\mathbf{a}|}, \text{ and } \cos\gamma = \frac{a_3}{|\mathbf{a}|}.$$

Note that

$$\frac{\mathbf{a}}{|\mathbf{a}|} = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle.$$

Exercise: Find the direction angles of $\mathbf{a} = \langle 1, 2, 3 \rangle$.

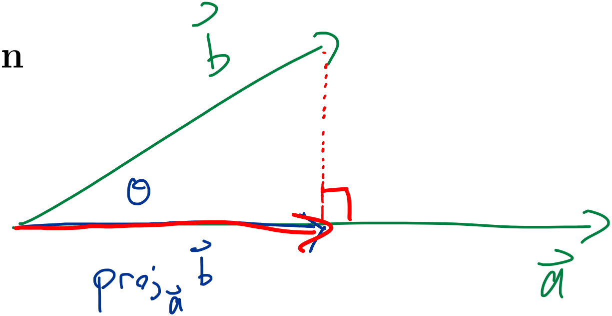
$$\cos\alpha = \frac{a_1}{|\vec{a}|} = \frac{1}{\sqrt{1^2+2^2+3^2}} = \frac{1}{\sqrt{14}}$$

$$\cos\beta = \frac{a_2}{|\vec{a}|} = \frac{2}{\sqrt{14}}$$

$$\cos\gamma = \frac{a_3}{|\vec{a}|} = \frac{3}{\sqrt{14}}$$

Orthogonal projection

$$|\operatorname{proj}_{\vec a}\vec b| = \operatorname{comp}_{\vec a}\vec b$$



$$\cos\theta = \frac{\operatorname{comp}_{\vec a}\vec b}{|\vec b|}$$

$$\cos\theta = \frac{\vec a\cdot\vec b}{|\vec a|\,|\vec b|} = \frac{\operatorname{comp}_{\vec a}\vec b}{|\vec b|}$$

$$\boxed{\operatorname{comp}_{\vec a}\vec b = \frac{\vec a\cdot\vec b}{|\vec a|}}$$

$$\mathrm{proj}_{\vec{a}}\vec{b} = \left(\mathrm{comp}_{\vec{a}}\vec{b}\right)\frac{\vec{a}}{|\vec{a}|}$$

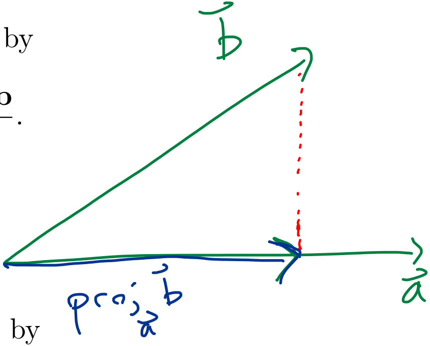
$$= \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)\frac{\vec{a}}{|\vec{a}|}$$

$$\mathrm{proj}_{\vec{a}}\vec{b} = \boxed{\frac{(\vec{a}\cdot\vec{b})}{|\vec{a}|^2}}\vec{a}$$

Orthogonal projection

The **scalar projection** of $\mathbf{b}$ onto $\mathbf{a}$ is given by

$$\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}.$$



The **vector projection** of $\mathbf{b}$ onto $\mathbf{a}$ is given by

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}.$$

Exercise: Find the scalar and vector projection of $\mathbf{b} = \langle 1, 1, 2 \rangle$ onto $\mathbf{a} = \langle -2, 3, 1 \rangle$.

$$\text{proj}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \vec{a}$$

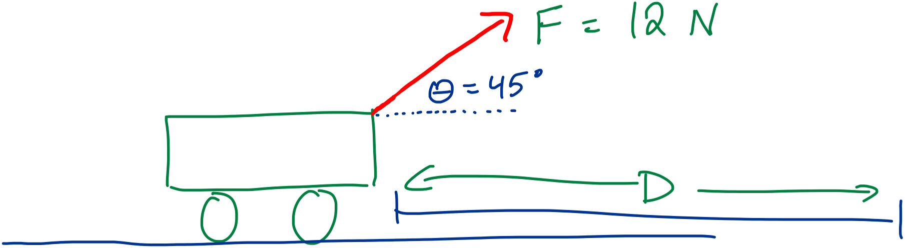
$$\vec{a} \cdot \vec{b} = (1)(-2) + (1)(3) + 2(1) = -2 + 3 + 2 = 3$$

$$|\vec{a}| = \sqrt{(-2)^2 + 3^2 + 1^2} = \sqrt{14}, \quad |\vec{a}|^2 = 14$$

$$\text{proj}_{\vec{a}} \vec{b} = \frac{3}{14} \langle -2, 3, 1 \rangle$$

$$\text{comp}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} = \frac{3}{\sqrt{14}} = \left| \text{proj}_{\vec{a}} \vec{b} \right|$$

Exercise: Work done pulling wagon.



$$\text{Distance} = D\hat{i} = \langle D, 0\rangle$$

$$\text{Force} = \vec{F}$$

$$\text{Work Done} = (D\hat{i})\cdot\vec{F} = |D\hat{i}|\,|\vec{F}|\cos\theta$$

$$= D(12)\cos(45^\circ)$$

$$= \frac{12P}{\sqrt{2}}$$