Math 1272: Calculus II
12.4 The Cross Product

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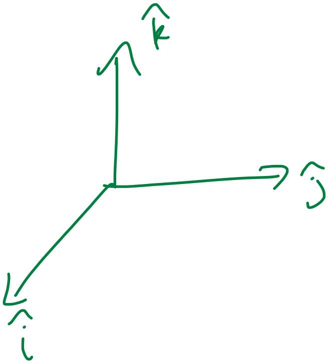
The cross product

Definition v1: The cross product $\mathbf{a} \times \mathbf{b}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$ is a vector orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

- Orientation?
- Length?

The cross product

Definition v2: The cross product $\mathbf{a} \times \mathbf{b}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$ is a vector orthogonal to both $\mathbf{a}$ and $\mathbf{b}$ with direction given by the **right hand rule**.



$\hat{i} \times \hat{j} = \hat{k}$ (not $-\hat{k}$)

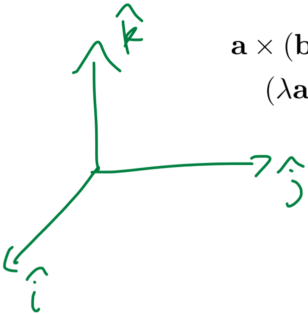
$\hat{j} \times \hat{i} = ?$ $(-\hat{k})$

The cross product

Definition v3: For the standard basis vectors

$$\begin{aligned} \mathbf{i} \times \mathbf{j} &= \mathbf{k} & \qquad \mathbf{j} \times \mathbf{i} &= -\mathbf{k} \\ \mathbf{i} \times \mathbf{k} &= -\mathbf{j} & \qquad \mathbf{k} \times \mathbf{i} &= \mathbf{j} \\ \mathbf{j} \times \mathbf{k} &= \mathbf{i} & \qquad \mathbf{k} \times \mathbf{j} &= -\mathbf{i}. \end{aligned}$$

We also want some nice properties



$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

$$(\lambda \mathbf{a}) \times \mathbf{b} = \lambda(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (\lambda \mathbf{b}).$$

Derivation of cross product

$$\vec{a} = \langle a_1, a_2, a_3 \rangle$$

$$\vec{b} = \langle b_1, b_2, b_3 \rangle$$

$$\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$$

$$\vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$$

$$\vec{a} \times \vec{b} = (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) \times (b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k})$$

$$= (a_1 \hat{i}) \times (b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k})$$

$$+ (a_2 \hat{j}) \times (b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k})$$

$$+ (a_3 \hat{k}) \times (b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k})$$

$$= \underbrace{\cancel{a_1 b_1 (\hat{i} \times \hat{i})}}_{0} + a_1 b_2 \underbrace{(\hat{i} \times \hat{j})}_{\hat{k}} + \underbrace{\cancel{a_1 b_3 (\hat{i} \times \hat{k})}}_{-\hat{j}}$$

$$+ a_2 b_1 \underbrace{(\hat{j} \times \hat{i})}_{-\hat{k}} + \underbrace{\cancel{a_2 b_2 (\hat{j} \times \hat{j})}}_{0} + \underbrace{\cancel{a_2 b_3 (\hat{j} \times \hat{k})}}_{\hat{i}}$$

$$+ \underbrace{\cancel{a_3 b_1 (\hat{k} \times \hat{i})}}_{\hat{j}} + \underbrace{\cancel{a_3 b_2 (\hat{k} \times \hat{j})}}_{-\hat{i}} + \underbrace{\cancel{a_3 b_3 (\hat{k} \times \hat{k})}}_{0}$$

$$= (a_2 b_3 - a_3 b_2)\hat{i} + (a_3 b_1 - a_1 b_3)\hat{j}$$

$$+ (a_1 b_2 - a_2 b_1)\hat{k}$$

Determinants

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \qquad (2\times 2 \text{ determinant})$$

3x3 determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$+ a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) - a_{12}\left(a_{21}a_{33} - a_{23}a_{31}\right)$$

$$+ a_{13}\left(a_{21}a_{32} - a_{22}a_{31}\right)$$

Turns out $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$

then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= (a_2 b_3 - a_3 b_2)\hat{i} - (a_1 b_3 - a_3 b_1)\hat{j}$$

$$+ (a_1 b_2 - a_2 b_1)\hat{k}$$

Cross product

Definition: If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the cross product of $\mathbf{a}$ and $\mathbf{b}$ is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle.$$

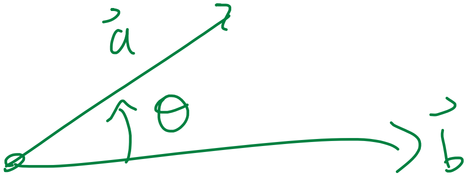
Properties of cross product

1. $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.
2. If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta.$$

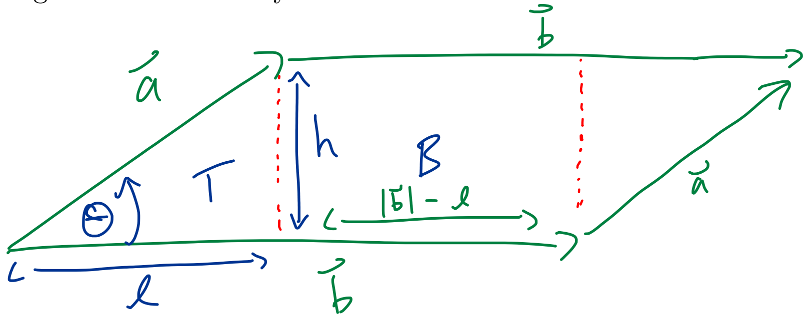
Ex

3. Two non-zero vectors are parallel if and only if $\mathbf{a} \times \mathbf{b} = 0$.



Areas of parallelograms

Fact: The length of the cross product $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$.



$$\sin\theta = \frac{h}{|\vec{a}|} \quad \rightsquigarrow \quad h = |\vec{a}| \sin\theta$$

$$\ell = |\vec{a}| \cos\theta$$

$$\cos\theta = \frac{\ell}{|\vec{a}|}$$

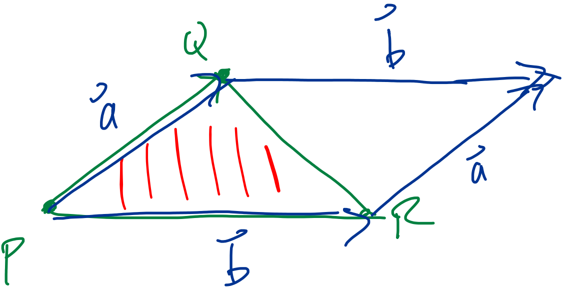
$$|\vec{a} \times \vec{b}| = 2\,\text{Area}(T) + \text{Area}(B)$$

$$= \cancel{\ell h} + (|\vec{b}| - \cancel{\ell)h}$$

$$= |\vec{b}|\,h$$

$$= |\vec{a}|\,|\vec{b}|\sin\theta. = |\vec{a} \times \vec{b}|.$$

Find the area of the triangle with vertices $P = (1, 4, 6)$, $Q = (-2, 5, -1)$, and $R = (1, -1, 1)$.



$\vec{a} = Q - P$

$\vec{b} = R - P$

$\vec{a} = \langle -3, 1, -7 \rangle$

$\vec{b} = \langle 0, -5, -5 \rangle$

Area of triangle $= \frac{1}{2}$ Area of parallelogram

$= \frac{1}{2} |\vec{a} \times \vec{b}|$

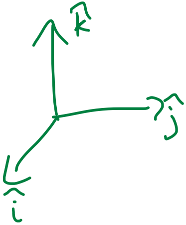
$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix}$$

$$= \hat{i}\left(-5 - (-5)(-7)\right) - \hat{j}\left((-3)(-5) - (-7)0\right)$$

$$+ \hat{k}\left((-3)(-5) - (1)0\right)$$

$$= -40\hat{i} - 15\hat{j} + 15\hat{k}$$

$$\text{Area} = \frac{1}{2}|\vec{a} \times \vec{b}| = \frac{1}{2}\sqrt{40^2 + 15^2 + 15^2} = \dots$$



Warning!!

Some laws of algebra do not hold for the cross product:

Non-commutative: $\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$.

$\hat{i} \times \hat{j} = \hat{k}$

$\hat{j} \times \hat{i} = -\hat{k}$

$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

Non-associate: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.

$\hat{i} \times (\hat{i} \times \hat{j}) = \hat{i} \times \hat{k} = -\hat{j}$

$(\hat{i} \times \hat{i}) \times \hat{j} = \hat{0} \times \hat{j} = 0$

Algebra with cross products

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

$$(\lambda \mathbf{a}) \times \mathbf{b} = \lambda(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (\lambda \mathbf{b})$$

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

$$(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$$

(*)
$$\boxed{\begin{aligned} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \\ \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}. \end{aligned}}$$

$$a \cdot (b \times c) = -a \cdot (c \times b)$$
$$= -(a \times c) \cdot b$$