Math 1272: Calculus II
12.5 Equations of lines and planes

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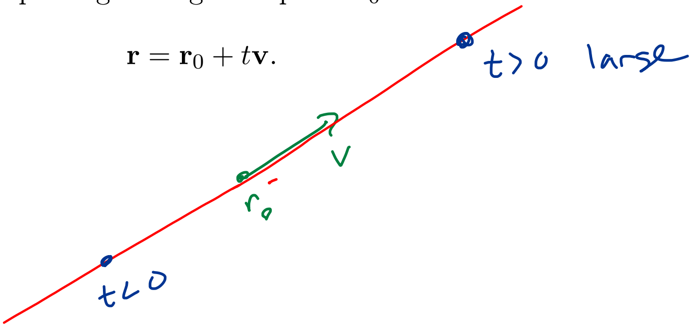
Lines

The equation for a line passing through the point $\mathbf{r}_0$ with direction $\mathbf{v}$ is

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}.$$



In parametric form

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$$

$$\vec{r}_0 = \langle x_0, y_0, z_0 \rangle, \quad \vec{v} = \langle a, b, c \rangle$$

$$\vec{r} = \vec{r}_0 + t\vec{v} = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$$

Find the vector and parametric equations for the line that passes through the point $(5, 1, 3)$ and is parallel to the vector $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$. Find 2 other points on the line.

$\vec{r}_0 = \langle 5, 1, 3 \rangle$, $\vec{v} = \langle 1, 4, -2 \rangle$

<u>vector</u>

$\vec{r} = \vec{r}_0 + t\vec{v} = \langle 5, 1, 3 \rangle + t\langle 1, 4, -2 \rangle$

<u>Parametric form</u>

$x = 5 + t$, $y = 1 + 4t$

$z = 3 - 2t$

<u>$t = 1$</u>

$\langle 6, 5, 1 \rangle$

<u>$t = -1$</u>

$\langle 4, -3, 5 \rangle$

We can eliminate t from the parametric equations

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct$$

to get **symmetric equations** for a line.

$$t = \frac{x - x_0}{a}, \quad t = \frac{y - y_0}{b}, \quad t = \frac{z - z_0}{c}$$

(3D) $$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

(2D) $$(y - y_0) = m(x - x_0)$$

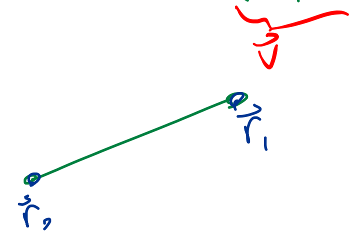
Find the parametric and symmetric eqautions for the line that passes through the points $A = (2, 4, -3)$ and $B = (3, -1, 1)$. At what point does the line intersect the xy-plane?

Line segment

The line segment from $\mathbf{r}_0$ to $\mathbf{r}_1$ is given by the vector equation

$$\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1, \qquad 0 \le t \le 1.$$

$$= \vec{r}_0 + t(\vec{r}_1 - \vec{r}_0)$$



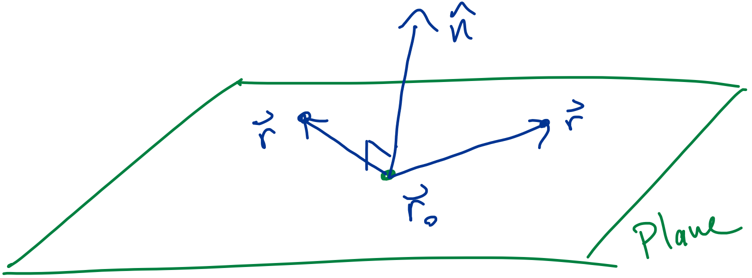
Planes

$$\vec{n}\cdot\vec{r} = \vec{n}\cdot\vec{r}_0$$

The equation for a plane is

$$\mathbf{n}\cdot(\mathbf{r}-\mathbf{r}_0)=0,$$

where $\mathbf{n}$ is a vector orthogonal to the plane, and $\mathbf{r}_0$ is a point on the plane.



Planes

The equation for a plane is

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0,$$

where $\mathbf{n}$ is a vector orthogonal to the plane, and $\mathbf{r}_0$ is a point on the plane.

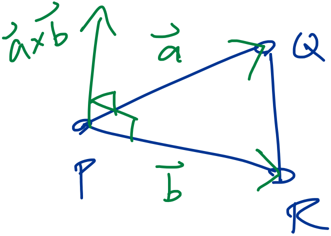
If $\mathbf{n} = \langle a, b, c \rangle$ and $\mathbf{r}_0 = (x_0, y_0, z_0)$, then the **scalar** equation for a plane is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0,$$

which has the form

$$ax + by + cz = d. \quad = ax_0 + by_0 + cz_0$$

Find an equation of the plane that passes through the points $P = (1, 3, 2)$, $Q = (3, -1, 6)$, and $R = (5, 2, 0)$.



$$\vec{a} = Q - P = \langle 2, -4, 4 \rangle$$

$$\vec{b} = R - P = \langle 4, -1, -2 \rangle$$

$$\vec{n} = \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix}$$

$$= (8 + 4)\hat{i} - (-4 - 16)\hat{j} + (-2 + 16)\hat{k}$$

$$= 12\hat{i} + 20\hat{j} + 14\hat{k} = \langle 12, 20, 14 \rangle$$

Vector equation is

$$\langle 12, 20, 14 \rangle \cdot \left(\vec{r} - (1,3,2) \right) = 0$$

If $\vec{r} = \langle x, y, z \rangle$ then

$$\langle 12, 20, 14 \rangle \cdot \langle x-1, y-3, z-2 \rangle = 0$$

Scalar form

$$12(x-1) + 20(y-3) + 14(z-2) = 0$$

$$12x + 20y + 14z = 12 + 60 + 28$$
$$= 100$$

Find the point at which the line with parametric equations $x = 2 + 3t$, $y = -4t$, $z = 5 + t$ intersects the plane $4x + 5y - 2z = 18$.

$$4(2+3t) + 5(-4t) - 2(5+t) = 18$$

$$8 + 12t - 20t - 10 - 2t = 18$$

$$-10t - 2 = 18$$

$$10t + 2 = -18$$

$$10t = -20 \longrightarrow \boxed{t = -2}$$

The point is $(-4, 8, 3)$.

$$x = 2 + 3(-2) = -4$$

$$y = -4(-2) = 8$$

$$z = 5 + (-2) = 3$$

Find the angle between the two planes $x + y + z = 1$ and $x - 2y + 3z = 1$, and find the symmetric equations for the line of intersection of the planes.

Distances

Find a formula for the distance D from a point $P_1 = (x_1, y_1, z_1)$ to the plane $ax + by + cz = d$.