Math 1272: Calculus II
7.3 Trigonometric Substitution

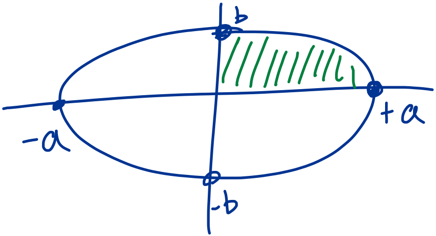
Instructor: Jeff Calder
Office: 538 Vincent
Email: jcalder@umn.edu

http://www-users.math.umn.edu/~jwcalder/1272S19

Warmup: Area of ellipse

Compute the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$



In 

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \longrightarrow y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$y = b\sqrt{1 - \frac{x^2}{a^2}}$$

$$y = \frac{b}{a}\sqrt{a^2 - x^2}$$

$$\text{Area} = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\, dx$$

$(*)\ \int_0^a \sqrt{a^2-x^2}\,dx$, If $x = a\sin\theta$

↳ means $\theta = \sin^{-1}\left(\frac{x}{a}\right)$

arcsin

$$a^2 - x^2 = a^2 - a^2\sin^2\theta$$

$$= a^2(1-\sin^2\theta) = a^2\cos^2\theta$$

So $\sqrt{a^2-x^2} = a\cos\theta$. $\frac{dx}{d\theta} = a\cos\theta$

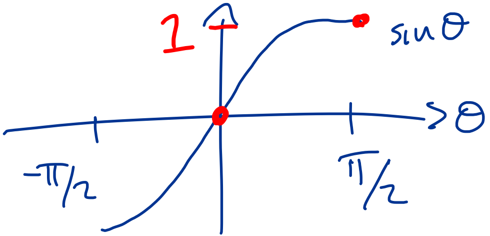
$$dx = a\cos\theta\, d\theta$$

$$(*) = \int_0^{\frac{\pi}{2}} a^2\cos^2\theta\, d\theta$$

Must have

$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$\int_0^a \sqrt{a^2-x^2}\,dx$$



$$= \int_0^{\pi/2} a^2\cos^2\theta\, d\theta$$

$$= \int_0^{\pi/2} \frac{a^2}{2}\left(1+\cos(2\theta)\right) d\theta$$

$$= \left[\frac{a^2}{2}\theta + \frac{a^2}{2}\cdot\frac{1}{2}\sin(2\theta)\right]_0^{\pi/2}$$

$$= \frac{a^2}{2}\frac{\pi}{2} + \frac{a^2}{4}\left(\sin(\pi) - \sin(0)\right) = \frac{\pi a^2}{4}$$

Area of ellipse $= \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$

$= \frac{4b}{a} \cdot \frac{\pi a^2}{4} = \pi ab$

Area of ellipse $= \pi ab$

Trigonometric substitution

When you see $\sqrt{a^2 - x^2}$, you can substitute

$$x = a \sin\theta \quad \text{with} \quad \frac{-\pi}{2} \le \theta \le \frac{\pi}{2},$$

and use the identity

$$1 - \sin^2\theta = \cos^2\theta.$$

Example 1. Find $\displaystyle\int \frac{\sqrt{4 - x^2}}{x^2}\,dx$.

Trigonometric substitution

When you see $\sqrt{a^2 + x^2}$, you can substitute

$$x = a \tan \theta \quad \text{with} \quad \frac{-\pi}{2} \le \theta \le \frac{\pi}{2},$$

and use the identity

$$1 + \tan^2 \theta = \sec^2 \theta.$$

Example 2. Find $\int \frac{1}{x^2\sqrt{x^2+16}}\, dx$. $a = 4$, $x = 4\tan\theta$.

$dx = 4\sec^2\theta \, d\theta$

$\sqrt{x^2+4^2} = \sqrt{4^2\tan^2\theta + 4^2} = 4\sqrt{1+\tan^2\theta}$

$= 4\sec\theta$

$x^2 = 16\tan^2\theta$

$$\int \frac{1}{x^2\sqrt{x^2+16}}\,dx = \int \frac{4\sec^2\theta}{64\sec\theta\tan^2\theta}\,d\theta$$

$$\frac{1}{\tan^2\theta} = \frac{\cos^2\theta}{\sin^2\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$= \frac{1}{16}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$

$$= \frac{1}{16}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

$u = \sin\theta$, $du = \cos\theta\, d\theta$

$$= \frac{1}{16}\int \frac{1}{u^2}\,du = \frac{1}{16}\int u^{-2}\,du$$

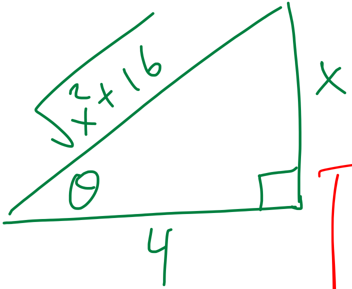
$u = \sin\theta$

$$= -\frac{1}{16} \cdot \frac{1}{u} + C$$

$x = 4\tan\theta$

$$= -\frac{1}{16\sin\theta} + C$$

$\tan\theta = \frac{x}{4}$



and $\sin\theta = \frac{x}{\sqrt{x^2+16}}$

$$= -\frac{\sqrt{x^2+16}}{16x} + C$$

Trigonometric substitution

When you see $\sqrt{x^2 - a^2}$, you can substitute

$$x = a \sec\theta \quad \text{with} \quad 0 \le \theta \le \frac{\pi}{2} \text{ or } \pi \le \theta \le \frac{3\pi}{2},$$

and use the identity

$$\sec^2\theta - 1 = \tan^2\theta.$$

Example 3. Evaluate $\displaystyle\int \frac{1}{\sqrt{x^2-9}}\,dx.$

$dx = a \tan\theta \sec\theta \, d\theta$

$$\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2} = a\sqrt{\sec^2\theta - 1}$$
$$= a\sqrt{\tan^2\theta}$$
$$= a\tan\theta$$

Here

$a = 3 \quad a^2 = 9$

$$\int \frac{1}{\sqrt{x^2-9}}\,dx = \int \frac{3\tan\theta \sec\theta\, d\theta}{3\tan\theta}$$

$$= \int \sec\theta\, d\theta$$

"trick" $$= \int \sec\theta \left(\frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}\right) d\theta$$

$$(*) = \int \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta}\, d\theta$$

Note: If $u = \sec\theta$, $du = \sec\theta\tan\theta\, d\theta$

If $u = \tan\theta$, $du = \sec^2\theta \, d\theta$

Set

$$u = \sec\theta + \tan\theta$$

$$du = (\tan\theta\sec\theta + \sec^2\theta)\,d\theta$$

$$(*) = \int \frac{1}{u}\,du$$

$$= \ln|u| + C$$

$$\int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$$

$$\int \frac{1}{\sqrt{x^2-9}}\,dx = \ln\left|\sec\theta + \tan\theta\right| + C$$

$x = 3\sec\theta$

$\sec\theta = \dfrac{x}{3}$

$\cos\theta = \dfrac{3}{x}$

$\tan\theta = \dfrac{\sqrt{x^2-9}}{3}$

$$= \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| + C$$