Math 1272: Calculus II
7.4 Integration of rational functions

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Integration of rational functions

We now study integration of rational functions

$$\int \frac{P(x)}{Q(x)}\,dx$$

where P, Q are polynomials

$$P(x) = a_n x^n + a_{n-1} x^{n-1} \cdots + a_1 x + a_0,$$

and

$$Q(x) = b_m x^m + b_{m-1} x^{m-1} \cdots + b_1 x + b_0$$

Integration of rational functions

Step 1: If $\deg(P) \geq \deg(Q)$ then divide (factor) so that

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

where $\deg(R) < \deg(Q)$.

Example 1. Compute $\int \frac{x^3-2x+1}{x+1}\,dx$.

Easy to integrate

$$x^3-2x+1 = (x+1)(x^2-x-1)+2$$

$$\boxed{\frac{x^3-2x+1}{x+1} = x^2-x-1+\frac{2}{x+1}}$$

Integration of rational functions

Step 2: Factor the denominator $Q(x)$ into linear and quadratic terms.

Example 2. Factor $Q(x) = x^4 - 9$.

$$Q(x) = (x^2-3)(x^2+3)$$

$$= \underbrace{(x-\sqrt{3})(x+\sqrt{3})}_{\text{Linear}}\underbrace{(x^2+3)}_{\text{irreducible quadratic}}$$

$b=0$
$a=1$
$c=3$

$$ax^2+bx+c \ : \ \boxed{b^2-4ac<0}$$

Integration of rational functions

By steps 1 and 2 we have

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)},$$

where $\deg(R) < \deg(Q)$, and we have factored Q.

Step 3: Use **Partial Fractions** to express $R(x)/Q(x)$ as a sum of partial fractions of the form

$$\frac{A}{(ax+b)^i} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^j}.$$

Step 4: Integrate term by term.

Integration of rational functions

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Step 4: Integrate term by term.

The procedure in step 3 depends on the factorization of Q.

Partial fractions

Case I: Q is a product of distinct linear factors

$$Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_kx + b_k).$$

In this case there exists $A_1, \ldots, A_k$ such that

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_2}{a_2x + b_2}.$$

Example 3. Integrate $\displaystyle\int \frac{x^2 - x + 2}{2x^3 + 3x^2 + x}\,dx.$

$$Q = 2x^3 + 3x^2 + x$$

$$Q(x) = x(2x^2 + 3x + 1)$$

$$= x\,\underset{x=-1}{\underline{(x+1)}}\,\underset{x=-\frac{1}{2}}{\underline{(2x+1)}}$$

$$x = \frac{-3 \pm \sqrt{9-8}}{4}$$

$$= \frac{1}{4}(-3 \pm 1)$$

$$= -\frac{1}{2}, -1$$

$$\frac{x^2-x+2}{2x^3+3x^2+x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{2x+1}$$

Expand + combine like terms

$$= \frac{A(x+1)(2x+1) + Bx(2x+1) + Cx(x+1)}{2x^3+3x^2+x}$$

$$x^2 - x + 2 = \underbrace{(2A+2B+C)}_{=1}x^2 + \overbrace{(3A+B+C)}^{=-1}x + \underbrace{A}_{=2}$$

$$2A + 2B + C = 1$$
$$3A + B + C = -1$$

$$A = 2$$

Plug in $A=2$

$$\begin{array}{r} 2B + C = -3 \\ -\ (B + C = -7) \\ \hline B = -3 + 7 = 4 \end{array}$$

$$C = -7 - B = -7 - 4 = -11$$

Easy to integrate

$$\boxed{\frac{x^2 - x + 2}{2x^3 + 3x^2 + x} = \frac{2}{x} + \frac{4}{x+1} - \frac{11}{2x+1}}$$

Partial fractions

Case II: Q is a product of linear factors, some of which are repeated:

$$Q(x) = (a_1x + b_1)^r (a_2x + b_2) \cdots (a_kx + b_k).$$

In the partial fraction expansion for the $(a_1x + b_1)^r$ term, we use

$$\frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \cdots + \frac{A_r}{(a_1x + b_1)^r}.$$

Example 4. Integrate $\displaystyle\int \frac{x}{x^3 - x^2 - x + 1}\, dx.$

$$Q(x) = x^3 - x^2 - x + 1$$

$$x = 1$$

$$Q(x) = (x+1)(x^2 - 2x + 1)$$

$$= (x+1)(x-1)^2$$

$$\frac{x}{\cancel{x^3-x^2-x+1}} = \left(\frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{(x-1)^2}\right)\frac{(x+1)(x-1)^2}{(x+1)(x-1)^2}$$

$$= \frac{A(x-1)^2 + B(x+1)(x-1) + C(x+1)}{\cancel{(x+1)(x-1)^2}}$$

$$x = A(x^2-2x+1) + B(x^2-1) + C(x+1)$$

$$= \underbrace{(A+B)}_{=0}x^2 + \underbrace{(C-2A)}_{=1}x + \underbrace{A-B+C}_{=0}$$

$A + B = 0$, $C - 2A = 1$, $A - B + C = 0$

$B = -A$, $C = 2A + 1$

$$A - B + C = 0$$

$$A + A + 2A + 1 = 0 \longrightarrow 4A = -1$$

$A = -\frac{1}{4}$

$B = \frac{1}{4}$

$C = \frac{1}{2}$

$$\frac{x}{x^3 - x^2 - x + 1} = \frac{-1/4}{x+1} + \frac{1/4}{x-1} + \frac{1/2}{(x-1)^2}$$

Integrate term-by-term.

Partial fractions

Case III: Q contains an irreducible non-repeated quadratic factor

$$ax^2 + bx + c.$$

The partial fraction expansion will have a term of the form

$$\frac{Ax + B}{ax^2 + bx + c}.$$

Example 5. Integrate $\displaystyle\int \frac{x^2 + x - 3}{x^3 + 9x}\,dx.$

$$Q(x) = x\underbrace{(x^2+9)}_{\text{irreducible quadratic}}$$

$$\frac{x^2+x-3}{\cancel{x^3+9x}} = \left(\frac{A}{x} + \frac{Bx+C}{x^2+9}\right)\frac{x(x^2+9)}{x(x^2+9)}$$

$$= \frac{A(x^2+9) + (Bx+C)x}{\cancel{x^3+9x}}$$

$$x^2 + x - 3 = \underbrace{(A+B)}_{=1}x^2 + \underbrace{C}_{=1}x + \underbrace{9A}_{=-3}$$

$$C = 1, \quad A = -\frac{1}{3}, \quad A + B = 1, \quad B = 1 - A$$

$$= 1 + \frac{1}{3}$$

$$= \frac{4}{3}$$

$$\frac{x^2+x-3}{x^3-9x} = \frac{-1/3}{x} + \frac{4/3\,x + 1}{x^2+9}$$

Integrate term by term

$$\int \frac{\frac{4}{3}x+1}{x^2+9}\,dx = \frac{4}{3}\underbrace{\int \frac{x}{x^2+9}\,dx} + \int \frac{1}{x^2+9}\,dx.$$

$$\int \frac{x}{x^2+9}\,dx = \int \frac{\frac{1}{2}\,du}{u+9} = \frac{1}{2}\int \frac{1}{u+9}\,du$$

$u = x^2,\ du = 2x\,dx$

$$= \frac{1}{2}\ln|u+9|$$

$$= \frac{1}{2}\ln|x^2+9|$$

$$\int \frac{1}{x^2+9}\,dx = \frac{1}{9}\int \frac{1}{(\frac{x}{3})^2+1}\,\boxed{dx}$$

$u = \frac{x}{3}$

$\boxed{du = \frac{1}{3}\,dx}$

$$= \frac{3}{9}\int \frac{1}{u^2+1}\,du = \frac{1}{3}\arctan\left(\frac{x}{3}\right)$$

Partial fractions

Case IV: Q contains a repeated irreducible quadratic factor

$$(ax^2+bx+c)^r.$$

The partial fraction expansion will have a term of the form

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_1x+B_1}{(ax^2+bx+c)^2}+\cdots+\frac{A_1x+B_1}{(ax^2+bx+c)^r}.$$

Example 6. Integrate $\displaystyle\int \frac{x^3-x-1}{x(x^2+1)^2}\,dx.$ $Q(x) = x(x^2+1)^2$

$$\frac{x^2-x-1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \boxed{\frac{Dx+E}{(x^2+1)^2}}$$

$= \cdots$ exercise.

How to find $\int \frac{x+1}{(x^2+1)^2}\,dx$?

$$= \underbrace{\int \frac{x}{(x^2+1)^2}\,dx} + \underbrace{\int \frac{1}{(x^2+1)^2}\,dx}_{(*)}$$

$$\left.\begin{array}{l} u = x^2 \\ du = 2x\,dx \end{array}\right\} = \int \frac{\frac{1}{2}\,du}{(u+1)^2} = -\frac{1}{2(u+1)} + C$$

$$(*) = \left.\begin{array}{l} \underline{\text{Trig sub.}} \\ x = \tan\theta \\ dx = \sec^2\theta\,d\theta \end{array}\right\} = \int \frac{\sec^2\theta\,d\theta}{(\tan^2\theta+1)^2}$$

$$= \int \frac{\sec^2\theta \, d\theta}{\sec^4\theta}$$

$$= \int \cos^2\theta \, d\theta \ldots$$