Math 1272: Calculus II
7.7 Approximate integration

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Approximate integration

Some functions are impossible to integrate exactly:

$$\int_0^1 e^{-x^2}\,dx \text{ or } \int_{-1}^1 \sqrt{1+x^3}\,dx.$$

Thus, we need some tools for approximating the integral with a computer.

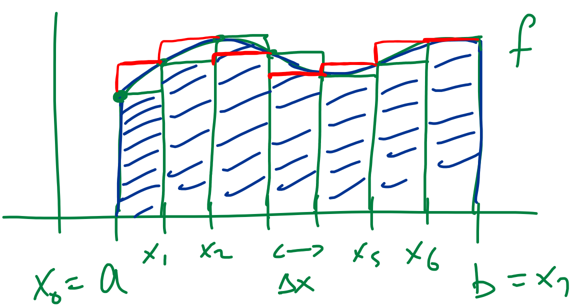
Idea: To approximate $\int_a^b f(x)\,dx$, split up the interval $a \leq x \leq b$ into small pieces, and approximate f by simpler functions (e.g., lines, parabolas, etc.) on each piece that you can integrate exactly.

Midpoint rule

— R_7

— L_7

$$\int_a^b f(x)\,dx$$



On interval $[x_i, x_{i+1}]$ $\qquad \Delta x = \dfrac{b-a}{7}$

approximate $f(x) \approx f(x_i)$

Then $\displaystyle\int_{x_i}^{x_{i+1}} f(x)\,dx \approx \int_{x_i}^{x_{i+1}} f(x_i)\,dx$

$$= f(x_i)\int_{x_i}^{x_{i+1}} dx = \Delta x\, f(x_i)$$

Left-point rule

$$\int_a^b f(x)\,dx \approx L_7 = \Delta x\, f(x_0) + \Delta x\, f(x_1) + \cdots$$

$$+ \cdots \Delta x\, f(x_5) + \Delta x\, f(x_6)$$

$$= \Delta x \big(f(x_0) + f(x_1) + \cdots + f(x_6) \big)$$

Right point rule

$f(x) \approx f(x_{i+1})$ on $[x_i, x_{i+1}]$

$$R_7 = \Delta x \big(f(x_1) + f(x_2) + \cdots + f(x_6) + f(x_7) \big)$$

Midpoint rule

The midpoint rule in general is

$$\int_a^b f(x)\,dx \approx M_n = \Delta x[f(\overline{x}_1) + f(\overline{x}_2) + \cdots + f(\overline{x}_n)],$$

where

$$\Delta x = \frac{b-a}{n}$$

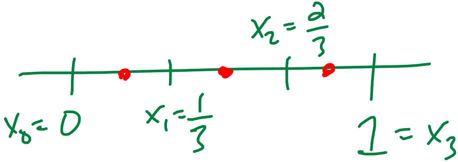
and

$$\overline{x}_i = \frac{1}{2}(x_{i-1} + x_i) = \text{midpoint of } [x_{i-1}, x_i].$$

Midpoint rule

Example 1. Use the midpoint rule with $n = 3$ to approximate $\int_0^1 3x^2\,dx$.

$$\Delta x = \frac{1-0}{3} = \frac{1}{3},$$



$$\int_0^1 3x^2\,dx \approx M_3 = \Delta x \left(f\left(\tfrac{1}{6}\right) + f\left(\tfrac{1}{2}\right) + f\left(\tfrac{5}{6}\right) \right)$$

$$= \frac{3}{3}\left(\frac{1}{6^2} + \frac{1}{2^2} + \frac{5^2}{6^2} \right)$$

$$= \frac{1}{36} + \frac{1}{4} + \frac{25}{36} = \frac{35}{36}$$

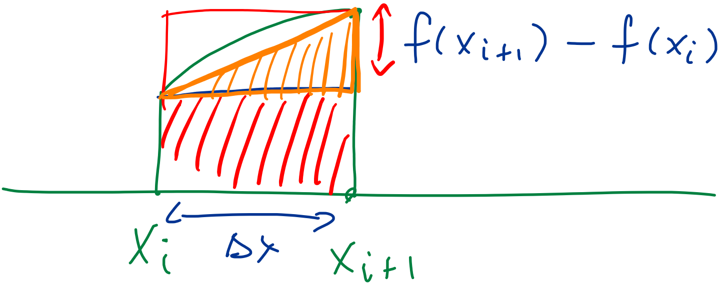
Trapezoid rule

$$L_5 = \Delta x \left(f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4) \right)$$

$$R_5 = \Delta x \left(f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) \right)$$

$$T_5 = \frac{R_5 + L_5}{2}$$

$$= \frac{\Delta x}{2} \left(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + f(x_5) \right)$$



Left pt $\approx \Delta x \, f(x_i)$, Right $\approx \Delta x \, f(x_{i+1})$

$$\underline{\text{Trapezoid}} = \frac{\Delta x \, f(x_i) + \Delta x \, f(x_{i+1})}{2}$$

$$= \underline{\Delta x \, f(x_i)} + \Delta x \, \frac{\left(f(x_{i+1}) - f(x_i)\right)}{2}$$

Compete the square

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 + c - \frac{b^2}{4}$$

Ex:

$$x^2 + 10x + 2 = (x+5)^2 + 2 - 5^2$$

$$= (x+5)^2 - 23$$

Trapezoid rule

The trapezoid rule in general is

$$\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)],$$

where

$$\Delta x = \frac{b-a}{n}$$

and

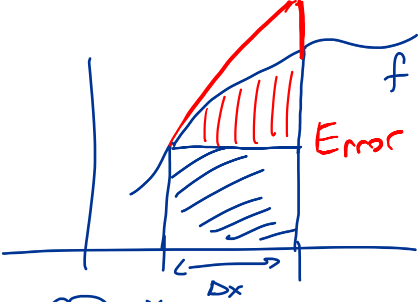
$$x_i = a + i\Delta x.$$

Trapezoid rule

Example 2. Use the trapezoid rule with $n = 3$ to approximate $\int_0^1 3x^2\,dx$.

Error bounds

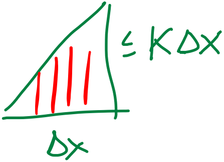
Left-point rule:



How large is the error (▨)?

Assume $|f'(x)| \le K$ for $a \le x \le b$

Error contained in triangle



$$\le \frac{bh}{2} = \frac{K\,\Delta x\,\Delta x}{2} = \frac{K}{2}\Delta x^2$$

we have n intervals, $\Delta x = \frac{b-a}{n}$

$$\text{Total error} \le \frac{K}{2}\Delta x^2 \cdot n$$

$$= \frac{K}{2}\left(\frac{b-a}{n}\right)^2 n$$

$$E_L = \frac{K(b-a)^2}{2n^1}$$

First order

Same for right-point rule

Error bounds for Trapezoid and Midpoint rules

Suppose $|f''(x)| \le K$ for all $a \le x \le b$. If E_T and E_M are the errors in the Trapezoid and Midpoint rules, then

$$|E_T| \le \frac{K(b-a)^3}{12n^2} \quad \text{and} \quad |E_M| \le \frac{K(b-a)^3}{24n^2}.$$

Second order accurate.

Example 3. How large should we make n so that the error in the Trapezoid rule for computing $\int_0^1 \sin(x)\,dx$ is less than 0.000001? Compare with the left or right point rules.

$$10^{-6} = \frac{K(b-a)^3}{12n^2} = \frac{1}{12n^2} \rightsquigarrow n^2 = \frac{10^6}{12}$$

$K=1$

$b-a=1$

$$n = \sqrt{\frac{10^6}{12}} \approx \frac{1000}{3}$$

left pt rule

$$\frac{1}{2n} = 10^{-6} \rightsquigarrow n = \frac{10^{6}}{2}$$

$$= 500{,}000$$

Simpson's rule

Simpson's rule approximates f by parabolas in each interval $[x_i, x_{i+1}]$.

In general, Simpson's rule is

$$\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\Big],$$

where n is even and

$$\Delta x = \frac{b-a}{n}.$$

Simpson's rule

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$\parallel$

Example 4. Use Simpson's rule with $n = 4$ to approximate $\int_0^1 3x^2\,dx$.

$$\Delta x = \frac{1-0}{4} = \frac{1}{4}, \quad x_0 = 0,\ x_1 = \frac{1}{4},\ x_2 = \frac{1}{2},\ x_3 = \frac{3}{4}$$

$$S_4 = \frac{\Delta x}{3}\left(f(0) + 4f\left(\tfrac{1}{4}\right) + 2f\left(\tfrac{1}{2}\right) + 4f\left(\tfrac{3}{4}\right) + f(1)\right)$$

$$= \Delta x\left(0 + 4\cdot\frac{1}{4^2} + 2\cdot\frac{1}{2^2} + 4\cdot\frac{3^2}{4^2} + 1\right)$$

$$= \frac{1}{4}\left(\frac{1}{4} + \frac{1}{2} + \frac{9}{4} + 1\right)$$

$$= \frac{1}{4}\left(\frac{1+2+9+4}{4}\right) = \frac{1}{4}\left(\frac{16}{4}\right) = \frac{4}{4} = 1$$

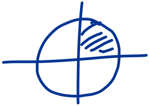
Error bounds for Simpson's rule

Suppose $|f^{(4)}(x)| \leq K$ for all $a \leq x \leq b$. If E_S is the error involved in Simpson's rule, then

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}.$$

4th order

Numerical examples

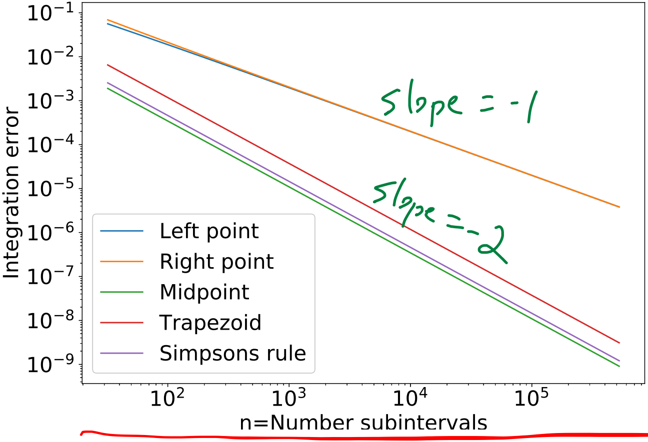


1. Area of a circle

$$E = \frac{C}{n^k}, \quad \log(E) = \log\left(\frac{C}{n^k}\right)$$

$$\pi = 4\int_0^1 \sqrt{1-x^2}\,dx$$

$$\log\left(\frac{C}{n^k}\right) = \log(C) - \log(n^k) = \log(C) - k\log(n)$$



$$\boxed{\log(E) = \log(C) - k\log(n)}$$

2. Integral of $1/x$:

$$1 = \int_1^e \frac{1}{x}\,dx.$$