Math 1272: Calculus II
8.1 Arclength

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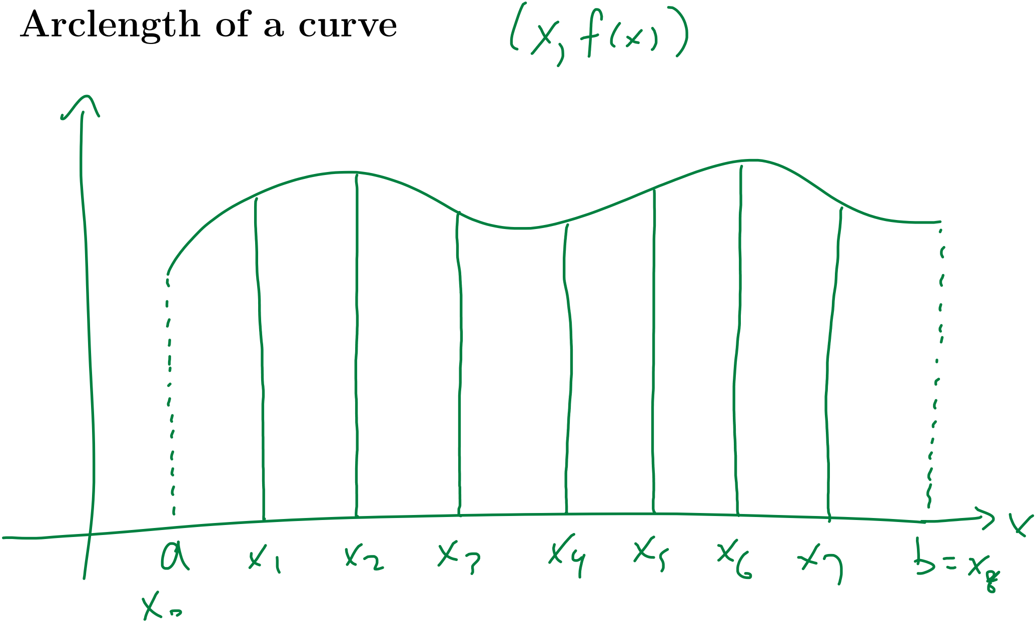
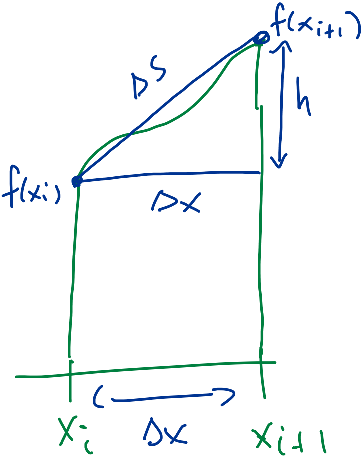
Arclength of a curve





$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{\Delta x}$$

$$h = f(x_{i+1}) - f(x_i)$$

$$\simeq \Delta x \, f'(x_i)$$

<u>Pythagorean theorem</u> $\Delta s^2 = \Delta x^2 + h^2$

$$= \Delta x^2 + \Delta x^2 [f'(x_i)]^2$$

$$= \Delta x^2 \left(1 + [f'(x_i)]^2\right)$$

Hence

$$\Delta s_i = \Delta x \sqrt{1 + [f'(x_i)]^2}$$

Sum over all n intervals, $\Delta x = \frac{b-a}{n}$

$$\underset{a \le x \le b}{\text{Arclength of } f} \approx \sum_{i=1}^{n} \Delta x \sqrt{1 + [f'(x_i)]^2}$$

As $n \to \infty$, $\Delta x \to 0$, get a Riemann sum for the integral

$$L(f) = \int_a^b \sqrt{1+f'(x)^2}\, dx$$

Arclength of f from $x=a$

to $x=b$.

Arclength of a curve

The *arclength* of a curve $y = f(x)$, $a \leq x \leq b$ is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

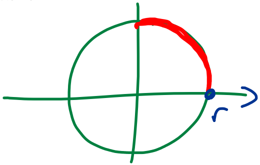
provided f' is continuous on $[a, b]$.

Arclength of a curve

Example 1. Show that the arclength of a circle of radius r

$$x^2 + y^2 = r^2$$

is $L = 2\pi r$.



$$L = 4\int_0^r \sqrt{1 + f'(x)^2}\, dx \qquad f(x) = y = \sqrt{r^2 - x^2}$$

$$f'(x) = \frac{1}{2}(r^2 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{r^2 - x^2}}$$

$$1 + f'(x)^2 = 1 + \frac{x^2}{r^2 - x^2}$$

$$= \frac{r^2 - x^2 + x^2}{r^2 - x^2} = \frac{r^2}{r^2 - x^2}$$

So

$$L = 4\int_0^r \sqrt{\frac{r^2}{r^2-x^2}}\, dx$$

arcsin

$\sin^{-1}(x)$

$\sin^2(x)$

$$= 4r\int_0^r \frac{1}{\sqrt{r^2-x^2}}\, dx$$

$$= 4r \arcsin\left(\frac{x}{r}\right)\Big]_0^r$$

$$= 4r\left(\arcsin(1) - \arcsin(0)\right)$$

$$= 4r\left(\frac{\pi}{2} - 0\right) = 2\pi r$$

Arclength of a curve

We can also compute the length of curves $x = g(y)$ for $c \leq y \leq d$ by

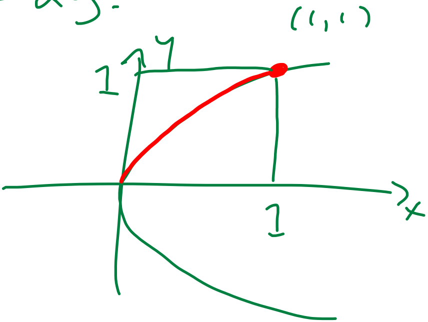
$$L = \int_a^b \sqrt{1 + [g'(y)]^2}\, dy.$$

Example 2. Find the arclength of the parabola $y^2 = x$ from $(0, 0)$ to $(1, 1)$.

Here $g(y) = y^2$, $g'(y) = 2y$.

$$L = \int_0^1 \sqrt{1 + 4y^2}\, dy$$

$$= 2\int_0^1 \sqrt{\frac{1}{4} + y^2}\, dy$$



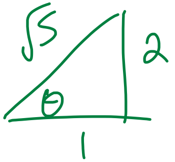
Trig sub $y = \frac{1}{2}\tan\theta$

and $dy = \frac{1}{2}\sec^2\theta \, d\theta$

$$\frac{1}{4} + y^2 = \frac{1}{4} + \frac{1}{4}\tan^2\theta$$

$$= \frac{1}{4}\left(1 + \tan^2\theta\right) = \frac{\sec^2\theta}{4}$$

$$L = 2\int_0^{\tan^{-1}(2)} \frac{\sec\theta}{2} \cdot \frac{1}{2}\sec^2\theta \, d\theta$$



Limits $\tan\theta : 0 \rightarrow 2$

$\theta : 0 \rightarrow \tan^{-1}(2)$

$$L = \frac{1}{2}\int_0^{\tan^{-1}(2)} \sec^3\theta \, d\theta.$$

$$\int u\,dv = uv - \int v\,du$$

$$\int \sec^3\theta \, d\theta = \sec\theta\tan\theta - \int \tan^2\theta \sec\theta \, d\theta$$

$$= \sec\theta\tan\theta - \int (\sec^2\theta - 1)\sec\theta \, d\theta$$

$$= \sec\theta\tan\theta - \int \sec^3\theta \, d\theta + \int \sec\theta \, d\theta$$

$u = \sec\theta$

$dv = \sec^2\theta \, d\theta$

$du = \tan\theta\sec\theta \, d\theta$

$v = \tan\theta$

Add $\int \sec^3\theta \, d\theta$ to both sides

$$2\int \sec^3\theta\, d\theta = \sec\theta\tan\theta + \int \sec\theta\, d\theta$$

$$\int \sec^3\theta\, d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int \sec\theta\, d\theta$$

$$\int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$$

then

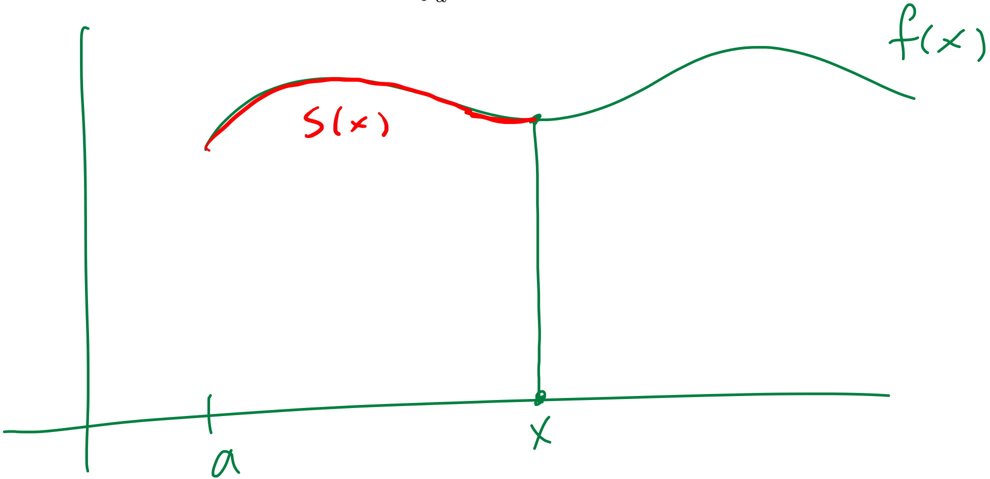
$$\int \sec^3\theta\, d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$$

Plug in limits..... $\theta = \tan^{-1}(2)$

Arclength function

The **arclength function** $s(x)$ of a curve $y = f(x)$ is defined by

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt.$$



$$S(x) = \int_a^x \sqrt{1 + f'(t)^2}\, dt$$

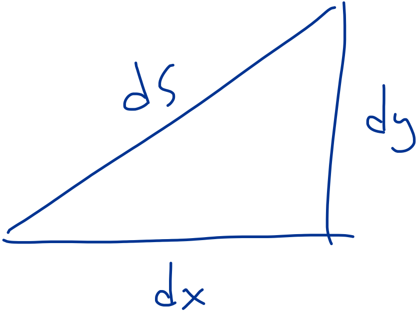
$$\frac{dS}{dx} = S'(x) = \sqrt{1 + f'(x)^2}$$

(Fundamental Theorem of Calculus)

Write $y = f(x)$, $\frac{dy}{dx} = f'(x)$

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

$$\boxed{ds^2 = dx^2 + dy^2}$$



$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Arclength function

The **arclength function** $s(x)$ satisfies

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

and

$$ds^2 = dx^2 + dy^2.$$

In this way, we have the compact formula for arclength

$$L = \int_a^b ds.$$

Arclength function

Example 3. Find the arclength function for the curve $y = x^2 - \frac{1}{8}\ln x$ taking $(1, 1)$ as the starting point.

$$\frac{dy}{dx} = 2x - \frac{1}{8x}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(2x - \frac{1}{8x}\right)^2$$

$$= 1 + 4x^2 - \frac{1}{2} + \frac{1}{64x^2}$$

$$= 4x^2 + \frac{1}{2} + \frac{1}{64x^2}$$

$$= \left(2x + \frac{1}{8x}\right)^2$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = 2x + \frac{1}{8x}$$

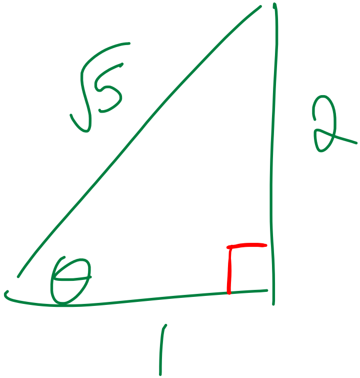
$$s(x) = \int_1^x 2t + \frac{1}{8t}\,dt$$

$$= t^2 + \frac{1}{8}\ln t \Big]_1^x$$

$$s(x) = x^2 + \frac{1}{8}\ln x - 1$$

what is $\sec(\tan^{-1}(2))$? $= \sqrt{5}$

$\theta = \tan^{-1}(2)$, $2 = \tan\theta$, $\sec(\theta)$



$$\cos\theta = \frac{1}{\sqrt{5}}$$

$$\sec\theta = \frac{1}{\cos\theta} = \sqrt{5}$$