Math 1272 Section 40: Midterm III Solutions

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Name:________________________________ Section Number:________________________________

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Instructions:

1. *Please don't turn over this page until you are directed to begin.*
2. There are 5 problems on this exam, and problem 5 has two parts.
3. There are 8 pages to the exam, including this page. All of them are one-sided. If you run out of room on the page you're working on, use the back of the page.
4. Please show all your work. Answers unsupported by an argument will get little credit.
5. Books, notes, calculators, cell phones, pagers, or other similar devices are not allowed during the exam. Please turn off cell phones for the duration of the exam. You may use the formula sheet attached to this exam.

Problem	Score
1	/10
2	/10
3	/10
4	/10
5	/10
Total:	/50

Some helpful formulas

$\cos x \cos y = \frac{1}{2}[\cos(x-y)+\cos(x+y)]$	$\tan^2 x + 1 = \sec^2 x$	$1 + \cot^2 x = \csc^2 x$
$\sin x \cos y = \frac{1}{2}[\sin(x-y)+\sin(x+y)]$	$\sin^2 x + \cos^2 x = 1$	$2\sin^2 x = 1 - \cos(2x)$
$\sin x \sin y = \frac{1}{2}[\cos(x-y)-\cos(x+y)]$	$2\sin x \cos x = \sin(2x)$	$2\cos^2 x = 1 + \cos(2x)$
$\int \sec x\, dx = \ln\lvert\sec x + \tan x\rvert$	$\int \tan x\, dx = \ln\lvert\sec x\rvert$	$\int \frac{1}{\sqrt{a^2-x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right)$
$\int \csc x\, dx = \ln\lvert\csc x - \cot x\rvert$	$\int \cot x\, dx = \ln\lvert\sin x\rvert$	$\int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$
$L = \int_a^b \sqrt{1+\left[\frac{dy}{dx}\right]^2}\, dx$	$\overline{x} = \frac{1}{m}\sum_{i=1}^{n} m_i x_i$	$\overline{x} = \frac{1}{A}\int_a^b x f(x)\, dx$
$S = \int_a^b 2\pi y\sqrt{1+\left[\frac{dy}{dx}\right]^2}\, dx$	$\overline{y} = \frac{1}{m}\sum_{i=1}^{n} m_i y_i$	$\overline{y} = \frac{1}{A}\int_a^b \frac{1}{2}[f(x)]^2\, dx$
$P(t) = \frac{M}{1+Ae^{-kt}}, \quad A = \frac{M-P_0}{P_0}$	$I(x) = e^{\int P(x)\, dx}$	$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1})$
$x = r\cos\theta, y = r\sin\theta$	$A = \int_a^b \frac{1}{2}[f(\theta)]^2\, d\theta$	$L = \int_a^b \sqrt{r^2+\left[\frac{dr}{d\theta}\right]^2}\, d\theta$
$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$	$L = \lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert}$	$c = \lim_{n\to\infty} \frac{a_n}{b_n}$
$L = \lim_{n\to\infty}\left\lvert\frac{a_{n+1}}{a_n}\right\rvert$	$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$	$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$
$\sin x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$	$\cos x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$	$\tan^{-1} x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$
$\ln(1+x) = \sum_{n=0}^{\infty}(-1)^{n-1}\frac{x^n}{n}$	$(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n$	$\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$

1. **(10 points)** Determine whether the series

$$\sum_{n=1}^{\infty} \frac{n^n}{n!}$$

converges or diverges.

Solution. Note that

$$n^n = \underbrace{n \cdot n \cdot n \cdot \cdots \cdot n \cdot n}_{n \text{ times}} \geq n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1 = n!.$$

Therefore $a_n = \frac{n^n}{n!} \geq 1$ and so $\lim_{n\to\infty} a_n \neq 0$. It follows that the series diverges. $\square$

2. **(10 points)** Determine whether the series

$$\sum_{n=1}^{\infty} \frac{\pi^n n^{103}}{n!}$$

converges or diverges.

Solution. We use the ratio test. We have

$$\begin{aligned} L := \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{\pi^{n+1}(n+1)^{103}}{(n+1)!} \frac{n!}{\pi^n n^{103}} \\ &= \lim_{n\to\infty} \frac{\pi^{n+1}(n+1)^{103}}{(n+1)!} \frac{n!}{\pi^n n^{103}} \\ &= \lim_{n\to\infty} \frac{\pi}{n+1} \left(1 + \frac{1}{n}\right)^{103} = 0. \end{aligned}$$

Therefore the series converges (absolutely). □

3. **(10 points)** Determine whether the series

$$\sum_{n=1}^{\infty} (-1)^n e^{-n}$$

converges absolutely, converges conditionally, or diverges.

Solution. Note that $a_n = (-1)^n e^{-n}$ and

$$|a_n| = e^{-n} = r^n$$

for $r = 1/e < 1$. Thus, $\sum |a_n|$ is a geometric series with $|r| < 1$, and hence converges. Thus, by definition, the series $\sum_{n=1}^{\infty} (-1)^n e^{-n}$ converges absolutely. We can also use the root test, since

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \frac{1}{e} < 1.$$

$\square$

4. **(10 points)** Determine whether the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

converges or diverges.

Solution. We use the integral test, which is valid since $a_n = \frac{1}{n(\ln n)^2}$ is positive and decreasing in n for $n \geq 2$. We compute

$$\begin{aligned}
\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx &= \lim_{T\to\infty} \int_2^{T} \frac{1}{x(\ln x)^2}\,dx \\
(u = \ln x, du = dx/x) &= \lim_{T\to\infty} \int_{\ln(2)}^{\ln(T)} \frac{1}{u^2}\,du \\
&= \lim_{T\to\infty} \left[-\frac{1}{u}\right]_{\ln(2)}^{\ln(T)} \\
&= \lim_{T\to\infty} \left[\frac{1}{\ln(2)} - \frac{1}{\ln(T)}\right] = \frac{1}{\ln(2)}.
\end{aligned}$$

Since the improper integral converges, the series converges as well, by the integral test. $\square$

5. Recall the hyperbolic trigonometric functions

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh(x) = \frac{e^x + e^{-x}}{2}.$$

(a) **(2 points)** Show that

$$\frac{d}{dx}\cosh(x) = \sinh(x) \quad \text{and} \quad \frac{d}{dx}\sinh(x) = \cosh(x).$$

Solution. This is a straightforward computation. □

(b) **(8 points)** Find the Maclaurin series for $f(x) = \cosh(x)$ and the interval of convergence. [**Hint**: While you may want to start by computing the first few terms of the series, a complete answer should give the general form of the series.]

Solution. After computing some derivatives, we find that $f^{(n)}(0) = 1$ when n is even, and $f^{(n)}(0) = 0$ when n is odd. Thus, the Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}.$$

To establish the interval of convergence, we use the Ratio test

$$\begin{aligned} L := \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \frac{|x^2|^{n+1}}{(2n+2)!} \frac{(2n)!}{|x^2|^n} \\ &= \lim_{n\to\infty} \frac{x^2}{(2n+2)(2n+1)} = 0. \end{aligned}$$

Hence, the series converges for all x and so the interval of convergence is $(-\infty, \infty)$. □