1.5 #3 (*) $(x_1+\cdots+x_n)^k = \sum_{|\alpha|=k} \binom{|\alpha|}{\alpha} x^\alpha$, $\binom{|\alpha|}{\alpha} := \frac{|\alpha|!}{\alpha!}$

Proof: By induction. The base case ~~A~~ = 2 is the binomial series

$$(x_1+x_2)^k = \sum_{n=0}^{k} \binom{k}{n} x_1^n x_2^{k-n} \qquad (1)$$

Assume (*) holds for all $k \in \mathbb{N}$. Consider a fixed $k \in \mathbb{N}$ and $n+1$ terms:

$$(x_1+\cdots+x_n+x_{n+1})^k = \sum_{i=0}^{k} \binom{k}{i} (x_1+\cdots+x_n)^i x_{n+1}^{k-i} \qquad \text{by (1)}$$

let $y=(x_1,\dots,x_n)$

$$= \sum_{i=0}^{k} \binom{k}{i} x_{n+1}^{k-i} \sum_{|\alpha|=i} \binom{|\alpha|}{\alpha} y^\alpha \qquad \text{by (*)}$$

$$\cancel{= \sum_{i=0}^{k} \binom{k}{i} \binom{|\alpha|}{\alpha}}$$

$$= \sum_{i=0}^{k} \sum_{|\alpha|=i} \binom{k}{i}\binom{|\alpha|}{\alpha} y^\alpha x_{n+1}^{k-i}$$

Let $\beta = (\alpha, k-i)$. Then $y^\alpha x_{n+1}^{k-i} = x^\beta$ and

$$\binom{k}{i}\binom{|\alpha|}{\alpha} = \frac{k!\,|\alpha|!}{(k-i)!\,i!\,\alpha!} = \frac{|\beta|!}{\beta!} = \binom{|\beta|}{\beta}$$

Since $|\beta| = |\alpha| + k - i = i + k - i = k$

and $\beta! = \alpha_1! \cdots \alpha_n! (k-i)! = \alpha!(k-i)!$

Hence
$$(x_1 + \cdots + x_{n+1})^k = \sum_{i=0}^{k} \sum_{\substack{|\beta| = k \\ \beta_{n+1} = k-i}} \binom{|\beta|}{\beta} x^\beta$$

$$= \sum_{|\beta| = k} \binom{|\beta|}{\beta} x^\beta$$

The proof is completed by induction. $\square$

3 Let $U \subset \mathbb{R}^n$ be open and bounded. Show that there does not exist a classical solution $u \in C^1(U) \cap C(\bar{U})$ of

$$\left.\begin{aligned} |Du| &= 1 \quad \text{in } U \\ u &= 0 \quad \text{on } \partial U \end{aligned}\right\} \quad (P)$$

Proof: Assume a classical solution u exists. Then u is continuous on the compact set $\bar{U}$. Therefore, u assumes its maximum value over $\bar{U}$ at some $x \in \bar{U}$. If $x \in U$ then

$$Du(x) = 0,$$

but by (P), $|Du(x)| = 1$. Hence $x \in \bar{U} \setminus U = \partial U$

Therefore $\max_{\bar{U}} u = \max_{\partial U} u = 0$

Hence $u \leq 0$ in U. By a similar argument, $u \geq 0$ in U. Therefore

$$u \equiv 0 \quad \text{in } \bar{U}.$$

This contradicts (P) (as $|Du| = 1$ in U). ☒

5| Prove that Laplace's equation $\Delta u = 0$ is rotation invariant.

Proof: Assume $\Delta u = 0$ in $\mathbb{R}^n$ (so $u \in C^2(\mathbb{R}^n)$) let O be an orthogonal $\underset{n \times n}{\wedge}$ matrix and define

$$v(x) := u(Ox) \quad \text{for } x \in \mathbb{R}^n.$$

Note $$v(x) = u\left(\sum_{i=1}^{n} O_{1i} x_i, \ldots, \sum_{i=1}^{n} O_{ni} x_i\right)$$

Hence $$v_{x_k} = \sum_{j=1}^{n} u_{x_j} O_{jk} \quad \text{and}$$

$$v_{x_k x_k} = \sum_{j=1}^{n} \sum_{i=1}^{n} u_{x_j x_i} O_{ik} O_{jk}$$

Therefore $$\Delta v = \sum_{k=1}^{n} v_{x_k x_k}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} u_{x_i x_j} \sum_{k=1}^{n} O_{ik} O_{jk}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} u_{x_i x_j} \underbrace{(OO^T)_{ij}}_{=1 \text{ if } i=j,\ 0 \text{ otherwise}}$$

$$= \sum_{i=1}^{n} u_{x_i x_i} = \Delta u = 0.$$ $\blacksquare$