Math 222A – Homework 2 (Due Friday Sept 18)

1. Evans: Section 2.5, Problem 3

2. (a) Give a direct proof that if $u \in C^2(U) \cap C(\overline{U})$ is harmonic within a bounded open set U, then
$$\max_{\overline{U}} u = \max_{\partial U} u.$$
[Hint: Define $u_\varepsilon := u + \varepsilon|x|^2$ for $\varepsilon > 0$ and show that u_ε cannot attain its maximum over $\overline{U}$ at an interior point.] [1]

   (b) Show that if $u \in C^2(U) \cap C(\overline{U})$ satisfies
$$-\Delta u + cu = 0 \quad \text{in} \quad U \qquad (c \geq 0),$$
then
$$\max_{\overline{U}} |u| = \max_{\partial U} |u|.$$

3. Evans: Section 2.5, Problem 5 (Problem 4 in 1st edition)

4. Let $u \in C^2(\overline{U})$ be harmonic in an open and bounded set U. Show that
$$\max_{\overline{U}} |Du| = \max_{\partial U} |Du|.$$

5. Let u be harmonic in an open and bounded set U. Use the mean value formula and divergence theorem to show that
$$|Du(x)| \leq \frac{n}{\operatorname{dist}(x, \partial U)} \left(\sup_U u - u(x) \right) \qquad \text{for all } x \in U.$$
In particular, if u is nonegative throughtout U, show that
$$|Du(x)| \leq \frac{n}{\operatorname{dist}(x, \partial U)} u(x) \qquad \text{for all } x \in U.$$

6. Evans: Section 2.5, Problem 6 (Problem 5 in 1st edition)

7. Recall Poisson's kernel for the half-space $\mathbb{R}^n_+$
$$K(x,y) = \frac{2x_n}{n\alpha(n)} \frac{1}{|x-y|^n} \qquad (x \in \mathbb{R}^n_+, \ y \in \partial\mathbb{R}^n_+).$$
Show that
$$\int_{\partial \mathbb{R}^n_+} K(x,y)\, dy = 1 \quad \text{for all } x \in \mathbb{R}^n_+. \tag{1}$$
[Hint: Let $A_n(x)$ denote the left hand side of (1). First show that $A_n(x) = A_n$ is independent of x. Then set $x = (0, \ldots, 0, x_n)$ and show that
$$A_n = \frac{2}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+} \frac{x_n}{|x-y|^n}\, dS(y) = \frac{2}{n\alpha(n)} \frac{2}{\pi} \int_0^\infty \frac{1}{1+x_n^2} \int_{\mathbb{R}^{n-1}} \frac{x_n}{(|y|^2 + x_n^2)^{n/2}}\, dy dx_n.$$
Then switch to polar coordinates.]

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[1] Section 2.5, Problem 4 in 2nd edition

8. Let $U \subset \mathbb{R}^n$ be open and bounded. Suppose that $u \in C(U)$ satisfies the mean value property

$$u(x) = \fint_{B(x,r)} u \, dy,$$

for all $B(x,r) \subset U$. Show that u is harmonic in U. [Hint: Use Poisson's formula for a ball and the maximum principle.]

9. Show that the uniform limit of a sequence of harmonic functions is harmonic.

10. Suppose that $u \in C(U)$ satisfies

$$\int_U u \Delta \varphi \, dx = 0$$

for all nonnegative $\varphi \in C^2(U)$ having compact support in U. Show that u is harmonic in U. [Hint: Use problem 9 and the standard mollifier.]

11. Evans: Section 2.5, Problem 7 (Problem 6 in 1st edition)