1] Modify the proof of the mean value formula to show that for $n \geq 3$

$$u(0) = \fint_{\partial B(0,r)} g\, dS + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}}\right) f\, dx$$

Provided

$$(P) \quad \begin{cases} -\Delta u = f, & \text{in } B^{0}(0,r) \\ u = g, & \text{on } \partial B^{0}(0,r). \end{cases}$$

Proof: As in the proof of the mean value formula, define

$$\phi(t) = \fint_{\partial B(0,t)} u\, dS$$

and compute

$$\phi'(t) = \frac{1}{n\alpha(n)t^{n-1}} \int_{B(0,t)} \Delta u(x)\, dx$$

$$= \frac{-1}{n\alpha(n)t^{n-1}} \int_{B(0,t)} f(x)\, dx \quad \text{by (P)}$$

If we define $\phi(0) = u(0)$ then ϕ is continuous on $[0,r]$ and differentiable on $(0,r)$ with

$$\phi(0) = u(0) \quad \text{and} \quad \phi(r) = \fint_{\partial B(0,r)} g\, ds \quad \text{by } (P)$$

Since $\int_{B(0,t)} f(x)\, dx \le C t^n$, $\phi'(t)$ is integrable and by the fundamental theorem of calculus

$$\phi(r) - \phi(0) = \int_0^r \phi'(t)\, dt$$

or

$$\fint_{\partial B(0,r)} g\, ds - u(0) = \int_0^r \frac{-1}{n\alpha(n)\, t^{n-1}} \int_{B(0,t)} f(x)\, dx\, dt$$

Therefore

$$u(0) = \fint_{\partial B(0,r)} g\, ds + \frac{1}{n\alpha(n)} \underbrace{\int_0^r \frac{1}{t^{n-1}} \int_{B(0,t)} f(x)\, dx\, dt}_{A}$$

Integrating in polar coordinates, we have

$$A = \int_0^r \frac{1}{t^{n-1}} \int_0^t \int_{\partial B(0,s)} f(y)\, dS(y)\, ds\, dt$$

We now integrate by parts to obtain $(n \geq 3)$

$$A = \int_0^r \frac{1}{(n-2)t^{n-2}} \int_{\partial B(0,t)} f(y)\, dS(y)\, dt$$

$$- \frac{1}{(n-2)\, t^{n-2}} \int_0^t \int_{\partial B(0,s)} f(y)\, dS(y)\, ds \Big|_{t=0}^{r}$$

Noting that $\left| \int_0^t \int_{\partial B(0,s)} f(y)\, dS(y)\, ds \right| \leq C \int_0^t s^{n-1}\, ds$

$$= C\, t^n$$

we see that $\lim\limits_{t \to 0^+} \frac{1}{(n-2)\, t^{n-2}} \int_0^t \int_{\partial B(0,s)} f(y)\, dS(y)\, ds = 0$

Therefore

$$\frac{1}{r^{n-2}}$$

$$A = \int_0^r \frac{1}{(n-2)\,t^{n-2}} \int_{\partial B(0,t)} f(y)\,dS(y)\,dt - \frac{1}{(n-2)\,\cancel{s}^{n-2}} \int_0^r \int_{\partial B(0,s)} f(y)\,dS(y)\,ds$$

$$= \frac{1}{n-2} \int_0^r \left(\frac{1}{t^{n-2}} - \frac{1}{r^{n-2}} \right) \int_{\partial B(0,t)} f(y)\,dS(y)\,dt$$

$$= \frac{1}{n-2} \int_{B(0,r)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right) f(x)\,dx \qquad \blacksquare$$

5. Let u be harmonic in an open and bounded set U. Show that

$$(1) \qquad |Du(x)| \leq \frac{n}{\operatorname{dist}(x, \partial U)} \left(\sup_U u - u(x) \right) \qquad \forall x \in U.$$

If u is nonnegative, show that

$$(2) \qquad |Du(x)| \leq \frac{n}{\operatorname{dist}(x, \partial U)} u(x) \qquad \forall x \in U.$$

Proof: Fix $x \in U$ and let $r > 0$ s.t.

$$B(x,r) \subset U$$

By the divergence theorem (and the fact that u_{x_i} is harmonic $\forall i$)

$$Du(x) = ⨍_{B(x,r)} Du(y)\, dy \qquad \text{← Mean value property for } u_{x_i}\ \forall i$$

$$= \frac{n}{r} ⨍_{\partial B(x,r)} u(y) N(y)\, dS(y) \qquad \text{← div thm}$$

N = outward unit normal

Since $\int_{\partial B(x,r)} N(y)\, dS(y) = 0$ we can write

$$Du(x) = \frac{n}{r} \fint_{\partial B(x,r)} (u(y) - u(x))\, N(y)\, dS(y)$$

Hence $$|Du(x)| = \frac{Du(x) \cdot Du(x)}{|Du(x)|}$$

$$= \frac{n}{r} \fint_{\partial B(x,r)} (u(y) - u(x)) \frac{N \cdot Du(x)}{|Du(x)|}\, dS(y).$$

By the mean value property $\fint_{\partial B(x,r)} u(y) - u(x)\, dS(y) = 0$

Therefore

$$|Du(x)| = \frac{n}{r} \fint_{\partial B(x,r)} (u(y) - u(x)) \underbrace{\left(\frac{N \cdot Du(x)}{|Du(x)|} + 1 \right)}_{\geq 0} dS(y)$$

$$\leq \frac{n}{r} \fint_{\partial B(x,r)} \left(\sup_U u - u(x) \right) \left(\frac{N \cdot Du(x)}{|Du(x)|} + 1 \right) dS(y)$$

$$= \frac{n}{r} \left(\sup_U u - u(x) \right) \fint_{\partial B(x,r)} \frac{N \cdot Du(x)}{|Du(x)|} + 1\, dS(y)$$

$$= \frac{n}{r} \left(\sup_U u - u(x) \right)$$

Sending $r \to \operatorname{dist}(x, \partial U)$ completes the proof of (1).

For (2), let $v(x) = -u(x)$. By (1)

$$|Du(x)| = |Dv(x)| \le \frac{n}{\operatorname{dist}(x,\partial U)}\left(\sup_U v - v(x)\right)$$

$$\le \frac{n}{\operatorname{dist}(x,\partial U)}\, u(x)$$

Since $\sup_U v \le 0$ and $-v(x) = u(x)$. $\boxed{///}$

u nonnegative

7] Recall Poisson's kernel for the half-space $\mathbb{R}^n_+$

$$K(x,y) = \frac{2x_n}{n\alpha(n)|x-y|^n}, \qquad x \in \mathbb{R}^n_+,\ y \in \partial\mathbb{R}^n_+$$

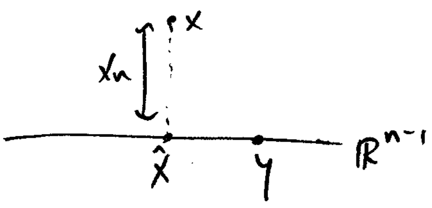
Show that

$$\int_{\partial\mathbb{R}^n_+} K(x,y)\,dy = 1 \quad \text{for all } x \in \mathbb{R}^n_+$$

Proof: Let $A_n(x) = \int_{\partial\mathbb{R}^n_+} K(x,y)\,dy$. We first claim that $A_n(x)$ is independent of x. To see this, convert to polar coordinates

$$A_n(x) = \int_0^\infty \int_{\partial B^{n-1}(\hat{x},r)} \frac{2x_n}{n\alpha(n)\,(r^2+x_n^2)^{n/2}}\,ds\,dr$$

$\hat{x} = (x_1, \ldots, x_{n-1})$



$|y-x|^2 = |y-\hat{x}|^2 + x_n^2$

$$= \int_0^\infty \frac{(n-1)\alpha(n-1)\,r^{n-2}\,2x_n}{n\alpha(n)\,(r^2+x_n^2)^{n/2}}\,dr$$

$$= \frac{2(n-1)\alpha(n-1)\,x_n}{n\alpha(n)} \int_0^\infty \frac{r^{n-2}}{(r^2+x_n^2)^{n/2}}\,dr$$

Make the change of variables $s = \frac{r}{x_n}$

Then

$$A_n(x) = \frac{2(n-1)\,\alpha(n-1)\,x_n}{n\alpha(n)} \int_0^\infty \frac{x_n^{n-2} s^{n-2}}{(x_n^2 s^2 + x_n^2)^{n/2}} x_n \, ds$$

$$= \frac{2(n-1)\,\alpha(n-1)}{n\alpha(n)} \int_0^\infty \frac{s^{n-2}}{(s^2+1)^{n/2}} \, ds$$

This establishes the claim. We can use complex residue theory to evaluate

$$\frac{2}{\pi} \int_0^\infty \frac{1}{1+x_n^2} \, dx_n = 1$$

Setting $x = (0, 0, \dots, 0, x_n)$ we have

$$A_n(x) = \frac{2}{n\alpha(n)} \int_{\partial \mathbb{R}^n_+} \frac{x_n}{|x-y|^n} \, dS(y), \quad |x-y|^2 = |y|^2 + x_n^2,$$

and $$A_n(x) = \frac{2}{\pi} \int_0^\infty \frac{A_n(x)}{1+x_n^2} \, dx_n$$

Therefore

$$A_n(x) = \frac{4}{n\alpha(n)\pi} \int_0^\infty \int_{\mathbb{R}^{n-1}} \frac{x_n}{(1+x_n^2)(|y|^2+x_n^2)^{n/2}} \, dy \, dx_n$$

$$= \frac{4}{n\alpha(n)\pi} \int_{\mathbb{R}^n_+} \frac{x_n}{(1+x_n^2)\,|x|^n} \, dx$$

Now switch to polar coordinates on the half plane $\mathbb{R}^n_+$:

$$A_n(x) = \frac{4}{n\alpha(n)\pi} \int_0^\infty \int_{\partial B_+(0,r)} \frac{y_n}{(1+y_n^2)\, r^n} \, dS(y) \, dr$$

Change of variables $z = \frac{y}{r}$, $dS(z) = \frac{dS(y)}{r^{n-1}}$

$$A_n(x) = \frac{4}{n\alpha(n)\pi} \int_0^\infty \int_{\partial B_+(0,1)} \frac{r z_n \, r^{n-1}}{(1+r^2 z_n^2)\, r^n} \, dS(z) \, dr$$

$$= \frac{4}{n\alpha(n)\pi} \int_{\partial B_+(0,1)} \int_0^\infty \frac{z_n}{1+r^2 z_n^2} \, dr \, dS(z)$$

Finally set $s = r z_n$ to find

$$A_n(x) = \frac{4}{n\alpha(n)\pi} \int_{\partial B_+(0,1)} \underbrace{\int_0^\infty \frac{1}{1+s^2}\,ds}_{=\frac{\pi}{2}}\, dS(z)$$

$$= \frac{2}{n\alpha(n)} \int_{\partial B_+(0,1)} dS(z) = \frac{2}{n\alpha(n)} \frac{|\partial B(0,1)|}{2}$$

$$= \frac{\cancel{2}}{n\alpha(n)} \cdot \frac{n\alpha(n)}{2} = 1$$

$\boxed{///}$

8. Let $U \subset \mathbb{R}^n$ be open and bounded. Suppose that $u \in C(U)$ satisfies the mean value property

$$u(x) = \fint_{B(x,r)} u\, dy$$

for all $B(x,r) \subset U$. Show that u is harmonic in U.

<u>Proof</u>: Fix $x^0 \in U$ and $R > 0$ s.t.

$$B(x^0, R) \subset U$$

Without loss of generality we may assume that $x^0 = 0$. Define

$$v(x) \overset{\text{def}}{=} \frac{R^2 - |x|^2}{n\alpha(n) R} \int_{\partial B(0,R)} \frac{u(y)}{|x-y|^n}\, dS(y) \quad , x \in B^0(0,R)$$

Then $v \in C^\infty(B^0(0,R))$, $\Delta v = 0$ in $B^0(0,R)$

and $\lim_{x \to x^0} v(x) = u(x^0)$ for all $x^0 \in \partial B^0(0,R)$

Hence v extends to a continuous function

$$v \in C(B(0,R)) \cap C^\infty(B^0(0,R))$$

that is harmonic in $B^0(0,R)$ and satisfies $v(x) = u(x)$ for all $x \in \partial B(0,R)$.

Set

$$w = v - u.$$

Clearly w satisfies the mean value property

$$w(x) = \fint_{B(x,r)} w(y)\,dy$$

for all $B(x,r) \subset B^0(0,R)$. Furthermore,

$$w(x) = v(x) - u(x) = 0 \quad \forall x \in \partial B(0,R).$$

By the maximum principle, $w \equiv 0$ in $B(0,R)$.

Thus u is harmonic in $B^0(0,R)$

This shows that u is harmonic in a neighborhood of every $x^0 \in U$

$\therefore$ u is harmonic in U. $\boxed{\,}$

9] Show that the uniform limit of a sequence of harmonic functions is harmonic.

Proof: Let $U \subset \mathbb{R}^n$ be open and let $u_n \in C^2(U)$ be a sequence of functions that are harmonic throughout U and converge uniformly on U to some function $u \in C(U)$.

Let $B(x,r) \subset U$. Then

$$u_n(x) = \fint_{B(x,r)} u_n(y)\, dy$$

Since $u_n \longrightarrow u$ uniformly on U (actually locally uniformly is suff.)

we have
$$u(x) = \fint_{B(x,r)} u(y)\, dy.$$

Hence u satisfies the mean value property on U and is harmonic by problem 8 $\boxtimes$

10 Suppose $u \in C(U)$ satisfies

$$\int_U u \, \Delta \varphi \, dx = 0 \qquad (*)$$

for all nonnegative $\varphi \in C^2(U)$ having compact support in U. Show that u is harmonic in U.

Proof: Let $\varepsilon > 0$ and

$$U_\varepsilon = \{ x \in U : \operatorname{dist}(x, \partial U) > \varepsilon \}.$$

Let η^ε be the standard mollifier and define

$$u^\varepsilon(x) = (\eta^\varepsilon * u)(x) \qquad \text{for } x \in U_\varepsilon$$

$$= \int_U \eta^\varepsilon(x-y) u(y) \, dy.$$

Since $u^\varepsilon \in C^\infty(U_\varepsilon)$ we can compute

$$\Delta u^\varepsilon(x) = \int_U \Delta \eta^\varepsilon(x-y)\, u(y)\, dy$$

Since $\eta^\varepsilon(x-\cdot)$ is nonnegative, smooth and has compact support in U, we can invoke $(\ast)$ to find that

$$\Delta u^\varepsilon(x) = \int_U \Delta \eta^\varepsilon(x-y)\, u(y)\, dy = 0$$

$\therefore$ u^ε is harmonic throughout U_ε.

Let $V \subset\subset U$ (compactly contained)

Then $u^\varepsilon \to u$ uniformly on V as $\varepsilon \to 0^+$

By problem 9, u is harmonic on V.

Since V was arbitrary, u is harmonic throughout U. $\boxed{///}$