Math 222A – Homework 3 (Due Sept 30)

1. (a) For $k \in \mathbb{N}$ and $\lambda > 0$, consider the $(2k)^{\text{th}}$-order linear PDE

$$(I - \lambda\Delta)^k u = f \quad \text{in } \mathbb{R}^n, \tag{1}$$

where $f \in L^2(\mathbb{R}^n)$. Use the Fourier Transform to formally derive the representation formula

$$u = S_{k,\lambda} * f, \tag{2}$$

where

$$\widehat{S_{k,\lambda}}(y) = \frac{1}{(2\pi)^{n/2}} \frac{1}{(1 + \lambda|y|^2)^k}. \tag{3}$$

(b) Show, formally, that

$$S_{k,\lambda}(x) = \frac{1}{(k-1)!(4\pi\lambda)^{n/2}} \int_0^\infty \frac{e^{-t-\frac{|x|^2}{4t\lambda}}}{t^{n/2-(k-1)}}\, dt. \tag{4}$$

[Hint: One way to do this is to first show that

$$S_{k,\lambda} = \underbrace{S_{1,\lambda} * \cdots * S_{1,\lambda}}_{k \text{ times}} = S_{k-1,\lambda} * S_{1,\lambda},$$

and then use induction on k.]

(c) Fix $\sigma > 0$ and set $\lambda(k) = \sigma^2/(2k)$. Show that

$$S_{k,\lambda(k)} \longrightarrow G_\sigma \quad \text{in } L^2(\mathbb{R}^n) \text{ as } k \to \infty,$$

where

$$G_\sigma(x) := \frac{1}{(2\sigma^2\pi)^{n/2}} e^{-\frac{|x|^2}{2\sigma^2}}.$$

It is worth thinking for a moment about the probabilistic interpretation of this limit (i.e, in the context of the central limit theorem).

2. Evans: Section 2.5, Problem 12 (Problem 10 in 1st edition)

3. Evans: Section 2.5, Problem 14 (Problem 12 in 1st edition)

4. Evans: Section 2.5, Problem 15 (Problem 13 in 1st edition)

5. Give a direct proof that if U is bounded and $u \in C^2_1(U_T) \cap C(\overline{U}_T)$ solves the heat equation, then

$$\max_{\overline{U}_T} u = \max_{\Gamma_T} u.$$

[Hint: Define $u_\varepsilon := u - \varepsilon t$ for $\varepsilon > 0$, and show that u_ε cannot attain its maximum over $\overline{U}_T$ at a point in U_T.]

6. Evans: Section 2.5, Problem 17 (Problem 14 in 1st edition)