Math 222A – Homework 3 Solutions

1. (a) For $k \in \mathbb{N}$ and $\lambda > 0$, consider the $(2k)^{\text{th}}$-order linear PDE

$$(I - \lambda\Delta)^k u = f \quad \text{in } \mathbb{R}^n, \tag{1}$$

where $f \in L^2(\mathbb{R}^n)$. Use the Fourier Transform to formally derive the representation formula

$$u = S_{k,\lambda} * f, \tag{2}$$

where

$$\widehat{S_{k,\lambda}}(y) = \frac{1}{(2\pi)^{n/2}} \frac{1}{(1 + \lambda|y|^2)^k}. \tag{3}$$

(b) Show, formally, that

$$S_{k,\lambda}(x) = \frac{1}{(k-1)!(4\pi\lambda)^{n/2}} \int_0^\infty \frac{e^{-t-\frac{|x|^2}{4t\lambda}}}{t^{n/2-(k-1)}}\, dt. \tag{4}$$

[Hint: One way to do this is to first show that

$$S_{k,\lambda} = \underbrace{S_{1,\lambda} * \cdots * S_{1,\lambda}}_{k \text{ times}} = S_{k-1,\lambda} * S_{1,\lambda},$$

and then use induction on k.]

Solution. The hint is useful to see how to derive the expression for $S_{k,\lambda}$. We can verify (4) more directly. Recall from class that

$$\mathcal{F}(e^{-\sigma|x|^2}) = \frac{1}{(2\sigma)^{n/2}} e^{-|x|^2/4\sigma}.$$

Let $g(x)$ denote the right hand side of (4). Then setting $\sigma = 1/4t\lambda$ we have

$$\begin{aligned}
\hat{g}(y) &= \frac{1}{(k-1)!(4\pi\lambda)^{n/2}} \int_0^\infty \frac{e^{-t}\mathcal{F}(e^{-\frac{|x|^2}{4t\lambda}})}{t^{n/2-(k-1)}}\, dt \\
&= \frac{1}{(k-1)!(2\pi)^{n/2}} \int_0^\infty t^{k-1} e^{-t(1+\lambda|y|^2)}\, dt.
\end{aligned}$$

Make the substitution $s = t(1 + \lambda|y|^2)$ to obtain

$$\hat{g}(y) = \frac{1}{(k-1)!(2\pi)^{n/2}} \frac{1}{(1+\lambda|y|^2)^k} \int_0^\infty s^{k-1} e^{-s}\, ds.$$

Since

$$\int_0^\infty s^{k-1} e^{-s}\, ds = \Gamma(k) = (k-1)!$$

we have

$$\hat{g}(y) = \frac{1}{(2\pi)^{n/2}} \frac{1}{(1+\lambda|y|^2)^k} = \widehat{S_{k,\lambda}}(y).$$

Therefore $S_{k,\lambda} = g$. □

(c) Fix $\sigma > 0$ and set $\lambda(k) = \sigma^2/(2k)$. Show that

$$S_{k,\lambda(k)} \longrightarrow G_\sigma \quad \text{in } L^2(\mathbb{R}^n) \ \text{ as } k \to \infty,$$

where

$$G_\sigma(x) := \frac{1}{(2\sigma^2\pi)^{n/2}} e^{-\frac{|x|^2}{2\sigma^2}}.$$

Solution. Let $S_k = S_{k,\lambda(k)}$ and notice that

$$\widehat{S_k}(y) = \frac{1}{(2\pi)^{n/2}} \left(1 + \frac{\sigma^2|y|^2}{2k}\right)^{-k}.$$

We suppose that $k \geq (n+1)/4$ so that $\widehat{S_k} \in L^2(\mathbb{R}^n)$. Using the identity

$$\left(1 + \frac{1}{x}\right)^x < e < \left(1 + \frac{1}{x}\right)^{x+1}$$

with $x = 2k/\sigma^2|y|^2$ and $y \neq 0$ yields

$$\widehat{S_k}(y) g_k(y) \leq \frac{e^{-\frac{\sigma^2|y|^2}{2}}}{(2\pi)^{n/2}} \leq \widehat{S_k}(y),$$

where

$$g_k(y) = \left(1 + \frac{\sigma^2|y|^2}{2k}\right)^{-\frac{\sigma^2|y|^2}{2}}.$$

The inequality above obviously holds when $y = 0$. Therefore

$$\|\widehat{S_k} - \widehat{G_\sigma}\|^2_{L^2(\mathbb{R}^n)} \leq \int_{\mathbb{R}^n} |\widehat{S_k}(y)|^2 |g_k(y) - 1|^2 \, dy.$$

By the Dominated Convergence Theorem, the right hand side tends to 0 as $k \to \infty$. Therefore

$$\widehat{S_k} \longrightarrow \widehat{G_\sigma} \quad \text{in } L^2(\mathbb{R}^n) \ \text{ as } k \to \infty.$$

The result follows by an application of Plancherel's Theorem

$$\|\widehat{S_k} - \widehat{G_\sigma}\|_{L^2(\mathbb{R}^n)} = \|S_k - G_\sigma\|_{L^2(\mathbb{R}^n)}.$$ □

It is worth thinking for a moment about the probabilistic interpretation of this limit (i.e, in the context of the central limit theorem).

2. Evans: Section 2.5, Problem 12 (Problem 10 in 1st edition)

3. Evans: Section 2.5, Problem 14 (Problem 12 in 1st edition)

Solution. You can use Duhamel's principle or the Fourier transform method to derive the solution

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} e^{-c(t-s)}\Phi(x-y,t-s)f(y,s)\,dy\,ds + \int_{\mathbb{R}^n} e^{-ct}\Phi(x-y,t)g(y)\,dy,$$

where $\Phi(x,t) = e^{-|x|^2/4t}/(4\pi t)^{n/2}$ is the fundamental solution of the heat equation. □

4. Evans: Section 2.5, Problem 15 (Problem 13 in 1st edition)

5. Give a direct proof that if U is bounded and $u \in C_1^2(U_T) \cap C(\overline{U}_T)$ solves the heat equation, then

$$\max_{\overline{U}_T} u = \max_{\Gamma_T} u.$$

[Hint: Define $u_\varepsilon := u - \varepsilon t$ for $\varepsilon > 0$, and show that u_ε cannot attain its maximum over $\overline{U}_T$ at a point in U_T.]

6. Evans: Section 2.5, Problem 17 (Problem 14 in 1st edition)